MA3486—Fixed Point Theorems and Economic Equilibria
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2.5. Intersections of Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \to Y$ be correspondences between X and Y. The *intersection* $\Phi \cap \Psi$ of the correspondences Φ and Ψ is defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$.

Proposition 2.20

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi\colon X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ be the correspondence defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

Proof

Let

$$W = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then W is the complement of the graph $\mathrm{Graph}(\Psi)$ of Ψ in $X \times Y$. The graph of Ψ is closed in $X \times Y$, by assumption. It follows that W is open in $X \times Y$.

Let $\mathbf{x} \in X$. The subset $\Psi(\mathbf{x})$ of Y is closed in Y, because the graph of the correspondence Ψ is closed. It follows from the compactness of $\Phi(\mathbf{x})$ that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ is a closed subset of the compact set $\Phi(\mathbf{x})$, and must therefore be compact. Thus the correspondence $\Phi \cap \Psi$ is compact-valued.

Now let $\mathbf{p}$ be a point of X, and let V be any open set in Y for which $\Phi(\mathbf{p}) \cap \Psi(\mathbf{p}) \subset V$. In order to prove that $\Phi \cap \Psi$ is upper hemicontinuous we must show that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \text{either } \mathbf{y} \in V \text{ or else } \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then U is the union of the subsets $X \times V$ and W of $X \times Y$, where both these subsets are open in $X \times Y$. It follows that U is open in $X \times Y$. Moreover if $\mathbf{y} \in \Phi(\mathbf{p})$ then either $\mathbf{y} \in \Phi(\mathbf{p}) \cap \Psi(\mathbf{p})$, in which case $\mathbf{y} \in V$, or else $\mathbf{y} \notin \Psi(\mathbf{p})$. It follows that $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Now it follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. Therefore there exists some positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $(\mathbf{x}, \mathbf{y}) \in X \times Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. Now if $(\mathbf{x}, \mathbf{y})$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ then $(\mathbf{x}, \mathbf{y}) \in U$ but $(\mathbf{x}, \mathbf{y}) \notin W$. It follows from the definition of U that $\mathbf{y} \in V$. Thus $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. $\blacksquare$

2.6. Berge's Maximum Theorem

Lemma 2.21

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < c \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) < c\}.$$

It follows from the continuity of the function f that U is open in $X \times Y$. It then follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. The result follows. ∎

Lemma 2.22

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) > c\},$$

and let

$$W = \{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\},$$

Let $\mathbf{p} \in W$. Then there exists $\mathbf{y} \in \Phi(\mathbf{p})$ for which $(\mathbf{p}, \mathbf{y}) \in U$. There then exist subsets W_X of X and W_Y of Y, where W_X is open in X and W_Y is open in Y, such that $\mathbf{p} \in W_X$, $\mathbf{y} \in W_Y$ and $W_X \times W_Y \subset U$ (see Lemma 2.5). There then exists some positive real number δ_1 such that $\mathbf{x} \in W_X$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_1$.

Now $\Phi(\mathbf{p}) \cap W_Y \neq \emptyset$, because $\mathbf{y} \in \Phi(\mathbf{p}) \cap W_Y$. It follows from the lower hemicontinuity of the correspondence Φ that there exists some positive real number δ_2 such that $\Phi(\mathbf{x}) \cap W_Y \neq \emptyset$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then there exists $\mathbf{y} \in \Phi(\mathbf{x})$ for which $\mathbf{y} \in W_Y$. But then $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$ and therefore $(\mathbf{x}, \mathbf{y}) \in U$, and thus $f(\mathbf{x}, \mathbf{y}) > c$. The result follows. ∎

Theorem 2.23 (Berge's Maximum Theorem)

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \to Y$ is both upper hemicontinuous and lower hemicontinuous. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof

Let $\mathbf{x} \in X$. Then $\Phi(\mathbf{x})$ is a non-empty compact subset of Y. It is thus a closed bounded subset of $\mathbb{R}^m$. It follows from the Extreme Value Theorem (Theorem 1.20) that there exists at least one point $\mathbf{y}^*$ of $\Phi(\mathbf{x})$ with the property that $f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Then $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^* \in M(\mathbf{x})$. Moreover

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}.$$

It follows from the continuity of f that the set $M(\mathbf{x})$ is closed in Y (see Corollary 1.17). It is thus a closed subset of the compact set $\Phi(\mathbf{x})$ and must therefore itself be compact.

Let some positive number ε be given. Then $f(\mathbf{p}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. It follows from Lemma 2.21 that

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X, and thus there exists some positive real number δ_1 such that $f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $\mathbf{y} \in \Phi(\mathbf{x})$ Then $m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$.

The correspondence $\Phi\colon X \rightrightarrows Y$ is also lower hemicontinuous. It therefore follows from Lemma 2.22 that there exists some positive real number δ_2 such that, given any $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_2$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ for which $f(\mathbf{x}, \mathbf{y}) > m(\mathbf{p}) - \varepsilon$. It follows that $m(\mathbf{x}) > m(\mathbf{p}) - \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$m(\mathbf{p}) - \varepsilon < m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $m\colon X \to \mathbb{R}$ is continuous on X.

It only remains to prove that the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous. Let

$$\Psi(\mathbf{x}) = \{\mathbf{y} \in Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then

$$\mathrm{Graph}(\Psi) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

Thus $\mathrm{Graph}(\Psi)$ is the preimage of zero under the continuous real-valued function that sends $(\mathbf{x}, \mathbf{y}) \in X \times Y$ to $f(\mathbf{x}, \mathbf{y}) - m(\mathbf{x})$. It follows that $\mathrm{Graph}(\Psi)$ is a closed subset of $X \times Y$.

Now $M(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ for all $\mathbf{x} \in X$, where the correspondence Φ is compact-valued and upper hemicontinuous and the correspondence Ψ has closed graph. It follows from Proposition 2.20 that the correspondence M must itself be both compact-valued and upper hemicontinuous. This completes the proof of Berge's Maximum Theorem. ∎