MA3486—Fixed Point Theorems and Economic Equilibria
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Proposition 2.17

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

Proof

Throughout this proof, let us say that the correspondence Φ satisfies the *constrained convergent subsequence criterion* if (and only if), given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

We must prove that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion if and only if it is compact-valued and upper hemicontinuous.

Suppose first that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Applying this criterion when $\mathbf{x}_j = \mathbf{p}$ for all positive integers j, we conclude that every infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\Phi(\mathbf{p})$ has a convergent subsequence, and therefore $\Phi(\mathbf{x})$ is compact.

Let

$$D = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}.$$

We show that D is closed in X. Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

be a sequence of points of D converging to some point of $\mathbf{p}$ of X. Then $\Phi(\mathbf{x}_j)$ is non-empty for all positive integers j, and therefore there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j. The constrained convergent subsequence criterion ensures that this infinite sequence in Y must have a subsequence that converges to some point of $\Phi(\mathbf{p})$. It follows that $\phi(\mathbf{p})$ is non-empty, and thus $\mathbf{p} \in D$.

Let $\mathbf{p}$ be a point of the complement of D. Then $\Phi(\mathbf{p}) = \emptyset$. There then exists $\delta > 0$ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\Phi(\mathbf{x}) \subset V$ for all open sets V in Y. It follows that the correspondence Φ is upper hemicontinuous at those points $\mathbf{p}$ for which $\Phi(\mathbf{p}) = \emptyset$.

Now consider the situation in which $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion and $\mathbf{p}$ is some point of X for which $\Phi(\mathbf{p}) \neq \emptyset$. Let $K = \Phi(\mathbf{p})$. Then K is a compact non-empty subset of Y. Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Suppose that there did not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It would then follow that there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $\mathbf{y}_j \notin V$.

Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, and thus the constrained convergent subsequence criterion satisfied by the correspondence Φ would ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. But then $\mathbf{q} \notin V$, because $\mathbf{y}_{k_j} \notin V$ for all positive integers j, and the complement $Y \setminus V$ of V is closed in Y. But $\Phi(\mathbf{p}) \subset V$, and $\mathbf{q} \in \Phi(\mathbf{p})$, and therefore $\mathbf{q} \in V$. Thus a contradiction would arise were there not to exist a positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus such a real number δ must exist, and thus the constrained convergent subsequence criterion ensures that the correspondence $\Phi \colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

It remains to show that any compact-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Let $\Phi\colon X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. It follows from Lemma 2.14 that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

be infinite sequences in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Then $\Phi(\mathbf{p})$ is non-empty, because

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X (see Lemma 2.14). Let $K = \Phi(\mathbf{p})$. Then K is compact, because $\Phi\colon X \rightrightarrows Y$ is compact-valued by assumption.

For each integer j let $d(\mathbf{y}_j, K)$ denote the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K. There then exists an infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K such that $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$. for all positive integers j. (Indeed if $d(\mathbf{y}_j, K) = 0$ then $\mathbf{y}_j \in K$, because the compact set K is closed, and in that case we can take $\mathbf{z}_j = \mathbf{y}_j$. Otherwise $2d(\mathbf{y}, K)$ is strictly greater than the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K, and we can therefore find $\mathbf{z}_j \in K$ with $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$.)

Now the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ ensures that $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. Indeed, given any positive real number ε, the set $B_Y(K, \varepsilon)$ of points of Y that lie within a distance ε of a point of K is an open set containing $\Phi(\mathbf{p})$. It follows from the upper hemicontinuity of Φ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(K, \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. But then $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and therefore $d(\mathbf{y}_j, K) < \varepsilon$ whenever $j \geq N$.

Now the compactness of K ensures that the infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K has a subsequence

$$\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \mathbf{z}_{k_3}, \ldots$$

that converges to some point $\mathbf{q}$ of K. Now $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$ for all positive integers j, and $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. It follows that $\mathbf{y}_{k_j} \to \mathbf{q}$ as $j \to +\infty$. Morever $\mathbf{q} \in \Phi(\mathbf{p})$. We have therefore verified that the constrained convergent subsequence criterion is satisfied by any correspondence $\Phi\colon X \rightrightarrows Y$ that is compact-valued and upper hemicontinuous. This completes the proof. ∎

Proposition 2.18

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof of Proposition 2.18 using Proposition 2.17

Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. Suppose that there did not exist any strictly positive real number δ with the property that $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any positive real number δ, there would exist points $\mathbf{x}$ of X and $\mathbf{y}$ of Y such that $|\mathbf{x} - \mathbf{p}| < \delta$, $\mathbf{y} \in \Phi(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \notin U$. Therefore there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively such that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ and $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$ for all positive integers j.

The correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous. Proposition 2.17 would therefore ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of Y converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. Now the complement of U in $X \times Y$ is closed in $X \times Y$, because U is open in $X \times Y$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$. It would therefore follow that $(\mathbf{p}, \mathbf{q}) \notin U$ (see Proposition 2.6). But this gives rise to a contradiction, because $\mathbf{q} \in \Phi(\mathbf{p})$ and $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. In order to avoid the contradiction, there must exist some positive real number δ with the property that with the property that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. The result follows. ∎

Remark

It should be noted that other results proved in this section do not necessarily generalize to correspondences $\Phi\colon X \rightrightarrows Y$ mapping the topological space X into an arbitrary topological space Y. For example all closed-valued upper hemicontinuous correspondences between metric spaces have closed graphs. The appropriate generalization of this result states that any closed-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ from a topological space X to a regular topological space Y has a closed graph. To interpret this, one needs to know the definition of what is meant by saying that a topological space is *regular*. A topological space Y is said to be *regular* if, given any closed subset F of Y, and given any point p of the complement $Y \setminus F$ of F, there exist open sets V and W in Y such that $F \subset V$, $p \in W$ and $V \cap W = \emptyset$. Metric spaces are regular. Also compact Hausdorff spaces are regular.

2.4. A Criterion characterizing Lower Hemicontinuity

Proposition 2.19

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

in X for which $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$.

Proof

First suppose that $\Phi\colon X \to Y$ is lower hemicontinuous at some point $\mathbf{p}$ of X. Let $\mathbf{q} \in \Phi(\mathbf{p})$, and let some positive number ε be given. Then the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centred on the point $\mathbf{q}$ is an open set in Y. It follows from the lower hemicontinuity of $\Phi\colon X \to Y$ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap B_Y(\mathbf{q}, \varepsilon)$ is non-empty whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < \varepsilon$. In particular, given any positive integer s, there exists some positive integer δ_s such that, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_s$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < 1/s$.

Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exist positive integers $k(1), k(2), k(3), \ldots$, where

$$k(1) < k(2) < k(3) < \cdots$$

such that $|\mathbf{x}_j - \mathbf{p}| < \delta_s$ for all positive integers j satisfying $j \geq k(s)$. There then exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $|\mathbf{y}_j - \mathbf{q}| < 1/s$ for all positive integers j and s satisfying $k(s) \leq j < k(s+1)$. Then $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. We have thus shown that if $\Phi\colon X \to Y$ is lower hemicontinuous at the point $\mathbf{p}$, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence in X converging to the point $\mathbf{p}$, and if $\mathbf{q} \in \Phi(\mathbf{p})$, then there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integer j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Next suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is not lower hemicontinuous at $\mathbf{p}$. Then there exists an open set V in Y such that $\Phi(\mathbf{p}) \cap V$ is non-empty but there does not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{p} - \mathbf{x}| < \delta$. Let $\mathbf{q} \in \Phi(\mathbf{p}) \cap V$. There then exists an infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap V = \emptyset$ for all positive integers j. It is not then possible to construct an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

The result follows. ∎