MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 6 (January 26, 2018)

David R. Wilkins

Proposition 2.10

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let K be a non-empty compact subset of Y, and let U be an subset in $X \times Y$ that is open in $X \times Y$. Let

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}$$

for all $\mathbf{y} \in Y$. Let $\mathbf{p}$ be a point of X with the property that $(\mathbf{p}, \mathbf{z}) \in U$ for all $\mathbf{z} \in K$. Then there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $d(\mathbf{y}, K) < \delta$.

Proof

Let

$$\tilde{K} = \{(\mathbf{p}, \mathbf{z}) : \mathbf{z} \in K\}.$$

Then $\tilde{K}$ is a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$. It follows from Proposition 2.9 that there exists some positive real number ε such that

$$B_{X \times Y}(\tilde{K}, \varepsilon) \subset U$$

where $B_{X \times Y}(\tilde{K}, \varepsilon)$ denotes that subset of $X \times Y$ consisting of those points $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ that lie within a distance ε of a point of $\tilde{K}$. Now a point $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ belongs to $B_{X \times Y}(\tilde{K}, \varepsilon)$ if and only if there exists some point $\mathbf{z}$ of K for which

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < \varepsilon^2.$$

Let $\delta = \varepsilon/\sqrt{2}$. If $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $d_Y(\mathbf{y}, K) < \delta$ then there exists some point $\mathbf{z}$ of K for which $|\mathbf{y} - \mathbf{z}| < \delta$. But then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < 2\delta^2 = \varepsilon^2,$$

and therefore $(\mathbf{x}, \mathbf{y}) \in U$, as required. $\blacksquare$

Proposition 2.11

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph $\mathrm{Graph}(\Phi)$ of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.

Proof

Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$.

Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$. Also $|\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required. ∎

Proposition 2.12

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof of Proposition 2.12 using Proposition 2.10

Let $\mathbf{p}$ be a point of X, let V be an open set satisfying $\Phi(\mathbf{p}) \subset V$, and let $K = Y \setminus V$. The compact set Y is closed and bounded in $\mathbb{R}^m$. Also K is closed in Y. It follows that K is a closed bounded subset of $\mathbb{R}^m$ (see Lemma 1.18). Let U be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then U is open in $X \times Y$, because $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$, because $\Phi(\mathbf{p}) \cap K = \emptyset$. It follows from Proposition 2.10 that there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in K$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\mathbf{y} \notin \Phi(\mathbf{x})$ for all $\mathbf{y} \in K$, and therefore $\Phi(\mathbf{x}) \subset V$, where $V = Y \setminus K$. Thus the correspondence Φ is upper hemicontinuous at $\mathbf{p}$, as required. ∎

Corollary 2.13

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the function $\varphi\colon X \to Y$ is continuous.

Proof

Let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\varphi)$, and therefore $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. The subset Y of $\mathbb{R}^m$ is compact. It therefore follows from Proposition 2.12 that the correspondence Φ is upper hemicontinuous. It then follows from Lemma 2.3 that the function $\varphi\colon X \to Y$ is continuous, as required. ∎

2.3. Compact-Valued Upper Hemicontinuous Correspondences

Lemma 2.14

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Proof

Given any open set V in Y, let

$$\Phi^{+}(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

It follows from the upper hemicontinuity of Φ that $\Phi^{+}(V)$ is open in X for all open sets V in Y (see Lemma 2.1). Now the empty set $\emptyset$ is open in Y. It follows that $\Phi^{+}(\emptyset)$ is open in X. But

$$\Phi^{+}(\emptyset) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset \emptyset\} = \{\mathbf{x} \in X : \Phi(\mathbf{x}) = \emptyset\}.$$

It follows that the set of point $\mathbf{x}$ in X for which $\Phi(\mathbf{x}) = \emptyset$ is open in X, and therefore the set of points $\mathbf{x} \in X$ for which $\Phi(\mathbf{x}) \neq \emptyset$ is closed in X, as required. ∎

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Given any subset S of X, we define the *image* $\Phi(S)$ of S under the correspondence Φ to be the subset of Y defined such that

$$\Phi(S) = \bigcup_{\mathbf{x} \in S} \Phi(\mathbf{x})$$

Lemma 2.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.

Proof

Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_{\mathbf{p}}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_{\mathbf{p}}$. Then $\Phi(\mathbf{p}) \subset U_{\mathbf{p}}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_{\mathbf{p}}$ in X such that $\Phi(\mathbf{x}) \subset U_{\mathbf{p}}$ for all $\mathbf{x} \in N_{\mathbf{p}}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_{\mathbf{p}}$ of open sets in Y covers $\Phi(N_{\mathbf{p}})$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows. ∎

Proposition 2.16

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \varepsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ε of some point of $\Phi(\mathbf{p})$.

Proof

Let $\Phi\colon X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi\colon X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$ (see Proposition 2.9). There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$ is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎