MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 5 (January 26, 2018)

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2.2. The Graph of a Correspondence

Let m and n be integers. Then the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ of the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimensions n and m is itself a Euclidean space of dimension $n + m$ whose Euclidean norm is characterized by the property that

$$|(\mathbf{x}, \mathbf{y})|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Lemma 2.4

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *and* $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ *be infinite sequences of points in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, and let* $\mathbf{p} \in \mathbb{R}^n$ *and* $\mathbf{q} \in \mathbb{R}^m$. *Then the infinite sequence*

$$(\mathbf{x}_1, \mathbf{y}_1), \, (\mathbf{x}_2, \mathbf{y}_2), \, (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ *to the point* $(\mathbf{p}, \mathbf{q})$ *if and only if the infinite sequence Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *converges to the point* $\mathbf{p}$ *and the infinite sequence* $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ *converges to the point* $\mathbf{q}$.

Proof

Suppose that the infinite sequence

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. But then

$$|\mathbf{x}_j - \mathbf{p}| < \varepsilon \quad \text{and} \quad |\mathbf{y}_j - \mathbf{q}| < \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$.

Conversely suppose that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_1$ and $|\mathbf{y}_j - \mathbf{q}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. Then

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. It follows that $(\mathbf{x}_j, \mathbf{y}_j) \to (\mathbf{p}, \mathbf{q})$ as $j \to +\infty$, as required. $\blacksquare$

Lemma 2.5

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let V be a subset of $X \times Y$. Then V is open in $X \times Y$ if and only if, given any point $(\mathbf{p}, \mathbf{q})$ of V, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$, there exist subsets W_X and W_Y of X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$, W_X is open in X, W_Y is open in Y and $W_X \times W_Y \subset V$.

Proof

Let V be a subset of $X \times Y$ and let $(\mathbf{p}, \mathbf{q}) \in V$, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$.

Suppose that V is open in $X \times Y$. Then there exists a positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in V$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < \delta^2.$$

Let

$$W_X = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \frac{\delta}{\sqrt{2}} \right\}$$

and

$$W_Y = \left\{ \mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| < \frac{\delta}{\sqrt{2}} \right\}$$

If $\mathbf{x} \in W_X$ and $\mathbf{y} \in W_Y$ then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < 2 \left(\frac{\delta}{\sqrt{2}} \right)^2 = \delta^2$$

and therefore $(\mathbf{x}, \mathbf{y}) \in V$. It follows that $W_X \times W_Y \subset V$.

Conversely suppose that there exist open sets W_X and W_Y in X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$ and $W_X \times W_Y \subset V$. Then there exists some positive real number δ such that $\mathbf{x} \in W_X$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and also $\mathbf{y} \in W_Y$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \delta$. If $(\mathbf{x}, \mathbf{y})$ is a point of $X \times Y$ that lies within a distance δ of $(\mathbf{p}, \mathbf{q})$ then $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{y} - \mathbf{q}| < \delta$, and therefore $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$. But $W_X \times W_Y \subset V$. It follows that the open ball of radius δ about the point $(\mathbf{p}, \mathbf{q})$ is wholly contained within the subset V of $X \times Y$. The result follows. $\blacksquare$

Proposition 2.6

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if

$$(\lim_{j\to\infty} \mathbf{x}_j, \lim_{j\to\infty} \mathbf{y}_j) \in G$$

for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.

Proof

Suppose that G is closed in $X \times Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be an infinite sequence in X converging to some point $\mathbf{p}$ of X and let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ be an infinite sequence in Y converging to a point $\mathbf{q}$ of Y, where $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j. We must prove that $(\mathbf{p}, \mathbf{q}) \in G$. Now the infinite sequence consisting of the ordered pairs $(\mathbf{x}_j, \mathbf{y}_j)$ converges in $X \times Y$ to $(\mathbf{p}, \mathbf{q})$ (see Lemma 2.4). Now every infinite sequence contained in G that converges to a point of $X \times Y$ must converge to a point of G, because G is closed in $X \times Y$ (see Lemma 1.10). It follows that $(\mathbf{p}, \mathbf{q}) \in G$.

Now suppose that G is not closed in $X \times Y$. Then the complement of G in $X \times Y$ is not open, and therefore there exists a point $(\mathbf{p}, \mathbf{q})$ of $X \times Y$ that does not belong to G though every open ball of positive radius about the point $(\mathbf{p}, \mathbf{q})$ intersects G. It follows that, given any positive integer j, the open ball of radius $1/j$ about the point $(\mathbf{p}, \mathbf{q})$ intersects G and therefore there exist $\mathbf{x}_j \in X$ and $\mathbf{y}_j \in Y$ for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $|\mathbf{y}_j - \mathbf{q}| < 1/j$ and $(\mathbf{x}_j, \mathbf{y}_j) \in G$. Then $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$ and therefore

$$(\lim_{j \to \infty} \mathbf{x}_j, \lim_{j \to \infty} \mathbf{y}_j) \notin G.$$

The result follows. ∎

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X and Y. The *graph* $\mathrm{Graph}(\varphi)$ of the function φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\varphi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} = \varphi(\mathbf{x})\}.$$

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. The *graph* $\mathrm{Graph}(\Phi)$ of the correspondence Φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\Phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Lemma 2.7

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that $\varphi\colon X \to Y$ is continuous. Then the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$.

Proof

Let $\psi\colon X \times Y \to Y$ be the function defined such that

$$\psi(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then $\mathrm{Graph}(\varphi) = \psi^{-1}(\{\mathbf{0}\})$, and $\{\mathbf{0}\}$ is closed in $\mathbb{R}^m$. It follows that $\mathrm{Graph}(\varphi)$ is closed in $X \times Y$ (see Corollary 1.17). ∎

Example

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then the graph $\mathrm{Graph}(f)$ of the function f satisfies $\mathrm{Graph}(f) = Z \cup H$, where

$$Z = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$$

and

$$H = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } xy = 1\}.$$

Each of the sets Z and H is a closed set in $\mathbb{R}^2$. It follows that $\mathrm{Graph}(f)$ is a closed set in $\mathbb{R}^2$. However the function $f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Lemma 2.8

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let S be a non-empty subset of X, and let

$$d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$$

for all $\mathbf{x} \in X$. Then the function sending $\mathbf{x}$ to $d(\mathbf{x}, S)$ for all $\mathbf{x} \in X$ is a continuous function on X.

Proof

Let $f(\mathbf{x}) = d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$ for all $\mathbf{x} \in X$.
Let $\mathbf{x}$ and $\mathbf{x}'$ be points of X. It follows from the Triangle Inequality that

$$f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{s}| \leq |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{s}|$$

for all $\mathbf{s} \in S$, and therefore

$$|\mathbf{x}' - \mathbf{s}| \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{s} \in S$. Thus $f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$ is a lower bound for the quantities $|\mathbf{x}' - \mathbf{s}|$ as $\mathbf{s}$ ranges over the set S, and therefore cannot exceed the greatest lower bound of these quantities. It follows that

$$f(\mathbf{x}') = \inf\{|\mathbf{x}' - \mathbf{s}| : \mathbf{s} \in S\} \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|,$$

and thus

$$f(\mathbf{x}) - f(\mathbf{x}') \leq |\mathbf{x} - \mathbf{x}'|.$$

Interchanging $\mathbf{x}$ and $\mathbf{x}'$, it follows that

$$f(\mathbf{x}') - f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}'|.$$

Thus

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{x}, \mathbf{x}' \in X$. It follows that the function $f\colon X \to \mathbb{R}$ is continuous, as required. $\blacksquare$

The multidimensional Heine-Borel Theorem (Theorem 1.23) ensures that a subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proposition 2.9

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbb{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ε with the property that $B_X(K, \varepsilon) \subset V$, where $B_X(K, \varepsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ε of some point of K.

Proof of Proposition 2.9 using the Extreme Value Theorem
Let $f\colon K \to \mathbb{R}$ be defined such that

$$f(\mathbf{x}) = \inf\{|\mathbf{z} - \mathbf{x}| : \mathbf{z} \in X \setminus V\}.$$

for all $\mathbf{x} \in K$. It follows from Lemma 2.8 that the function f is continuous on K.

Now $K \subset V$ and therefore, given any point $\mathbf{x} \in K$, there exists some positive real number δ such that the open ball of radius δ about the point $\mathbf{x}$ is contained in V, and therefore $f(\mathbf{x}) \geq \delta$. It follows that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$.

It follows from the Extreme Value Theorem for continuous real-valued functions on closed bounded subsets of Euclidean spaces (Theorem 1.20) that the function $f\colon K \to \mathbb{R}$ attains its minimum value at some point of K. Let that minimum value be ε. Then $f(\mathbf{x}) \geq \varepsilon > 0$ for all $\mathbf{x} \in K$, and therefore $|\mathbf{x} - \mathbf{z}| \geq \varepsilon > 0$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in X \setminus V$. It follows that $B_X(K, \varepsilon) \subset V$, as required. ∎

Example

Let

$$F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } xy \geq 1\}.$$

and let

$$V = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Note that if $(x, y) \in F$ then $x > 0$ and $y > 0$, because $xy = 1$. It follows that $F \subset V$. Also F is a closed set in $\mathbb{R}^2$ and V is an open set in $\mathbb{R}^2$. However F is not a compact subset of $\mathbb{R}^2$ because F is not bounded.

We now show that there does not exist any positive real number ε with the property that $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$, where $B_{\mathbb{R}^2}(F, \varepsilon)$ denotes the set of points of $\mathbb{R}^2$ that lie within a distance ε of some point of F. Indeed let some positive real number ε be given, let x be a positive real number satisfying $x > 2\varepsilon^{-1}$, and let $y = x^{-1} - \frac{1}{2}\varepsilon$. Then $y < 0$, and therefore $(x, y) \notin V$. But $(x, y + \frac{1}{2}\varepsilon) \in F$, and therefore $(x, y) \in B_{\mathbb{R}^2}(F, \varepsilon)$. This shows that there does not exist any positive real number ε for which $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$.