MA3486—Fixed Point Theorems and Economic Equilibria
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2. Correspondences and Hemicontinuity

2.1. Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A *correspondence* $\Phi\colon X \rightrightarrows Y$ assigns to each point $\mathbf{x}$ of X a subset $\Phi(\mathbf{x})$ of Y.

The *power set* $\mathcal{P}(Y)$ of Y is the set whose elements are the subsets of Y. A correspondence $\Phi\colon X \rightrightarrows Y$ may be regarded as a function from X to $\mathcal{P}(Y)$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the following definitions apply:–

- the correspondence $\Phi\colon X \to Y$ is said to be *non-empty-valued* if $\Phi(\mathbf{x})$ is a non-empty subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *closed-valued* if $\Phi(\mathbf{x})$ is a closed subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *compact-valued* if $\Phi(\mathbf{x})$ is a compact subset of Y for all $\mathbf{x} \in X$.

The multidimensional Heine-Borel Theorem (Theorem 1.23) ensures that the correspondence $\Phi\colon X \to Y$ is compact-valued if and only if $\Phi(\mathbf{x})$ is a closed bounded subset of $\mathbb{R}^m$ for all $\mathbf{x} \in X$.

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is upper hemicontinuous on X if it is upper hemicontinuous at each point of X.

Example

Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are upper hemicontinuous at x for all non-zero real numbers x. The correspondence F is also upper hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $F(0) \subset V$ then $[0,3] \subset V$ and therefore $F(x) \in V$ for all real numbers x.

However the correspondence G is not upper hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : \tfrac{1}{2} < y < \tfrac{5}{2}\}.$$

Then $G(0) \subset V$, but $G(x)$ is not contained in V for any positive real number x. Therefore there cannot exist any positive real number δ such that $G(x) \subset V$ whenever $|x| < \delta$.

Let

$$\mathrm{Graph}(F) = \{(x, y) \in \mathbb{R}^2 : y \in F(x)\}$$

and

$$\mathrm{Graph}(G) = \{(x, y) \in \mathbb{R}^2 : y \in G(x)\}.$$

Then $\mathrm{Graph}(F)$ is a closed subset of $\mathbb{R}^2$ but $\mathrm{Graph}(G)$ is not a closed subset of $\mathbb{R}^2$.

Example

Let S^1 be the unit circle in $\mathbb{R}^2$, defined such that

$$S^1 = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\},$$

let Z be the closed square with corners at $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$, so that

$$Z = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}.$$

Let $g_{(u,v)} \colon \mathbb{R}^2 \to \mathbb{R}$ be defined for all $(u, v) \in S^1$ such that

$$g_{(u,v)}(x, y) = ux + vy,$$

and let $\Phi \colon S^1 \rightrightarrows \mathbb{R}^2$ be defined such that, for all $(u, v) \in S^1$, $\Phi(u, v)$ is the subset of $\mathbb{R}^2$ consisting of the point of points of Z at which the linear functional $g_{(u,v)}$ attains its maximum value on Z.

Thus a point (x, y) of Z belongs to $\Phi(u, v)$ if and only if $g_{(u,v)}(x, y) \geq g_{(u,v)}(x', y')$ for all $(x', y') \in Z$. Then

$$\Phi(u, v) = \begin{cases} \{(1, 1)\} & \text{if } u > 0 \text{ and } v > 0; \\ \{(x, 1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v > 0; \\ \{(-1, 1)\} & \text{if } u < 0 \text{ and } v > 0; \\ \{(-1, y) : -1 \leq y \leq 1\} & \text{if } u < 0 \text{ and } v = 0; \\ \{(-1, -1)\} & \text{if } u < 0 \text{ and } v < 0; \\ \{(x, -1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v < 0; \\ \{(1, -1)\} & \text{if } u > 0 \text{ and } v < 0; \\ \{(1, y) : -1 \leq y \leq 1\} & \text{if } u > 0 \text{ and } v = 0. \end{cases}$$

It is a straightforward exercise to verify that the correspondence $\Phi \colon S^1 \rightrightarrows \mathbb{R}^2$ is upper hemicontinuous.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^+(V)$ the subset of X defined such that

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Lemma 2.1

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^+(V)$ is open in X.

Proof

First suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^+(V)$. Then $\Phi(\mathbf{p}) \subset V$. It then follows from the definition of upper hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^+(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^+(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^+(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \subset V$. Then $\Phi^+(V)$ is open in X and $\mathbf{p} \in \Phi^+(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^+(V).$$

Then $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$. The result follows. $\blacksquare$

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \emptyset$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is lower hemicontinuous on X if it is lower hemicontinuous at each point of X.

Example

Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are lower hemicontinuous at x for all non-zero real numbers x. The correspondence G is also lower hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $G(0) \cap V \neq \emptyset$ then $[1,2] \cap V \neq \emptyset$ and therefore $G(x) \cap V \neq \emptyset$ for all real numbers x.

However the correspondence F is not lower hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : 0 < y < \tfrac{1}{2}\}.$$

Then $F(0) \cap V \neq \emptyset$, but $F(x) \cap V = \emptyset$ for all negative real numbers x. Therefore there cannot exist any positive real number δ such that $F(x) \cap V \neq \emptyset$ whenever $|x| < \delta$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^{-}(V)$ the subset of X defined such that

$$\Phi^{-}(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \cap V \neq \emptyset\}.$$

Lemma 2.2

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^-(V)$ is open in X.

Proof

First suppose that $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^-(V)$. Then $\Phi(\mathbf{p}) \cap V$ is non-empty. It then follows from the definition of lower hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V$ is non-empty for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^-(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^-(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^{-}(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \cap V \neq \emptyset$. Then $\Phi^{-}(V)$ is open in X and $\mathbf{p} \in \Phi^{-}(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^{-}(V).$$

Then $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *continuous* at a point $\mathbf{p}$ of X if it is both upper hemicontinuous and lower hemicontinuous at $\mathbf{p}$. The correspondence Φ is continuous on X if it is continuous at each point of X.

Lemma 2.3

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\varphi\colon X \to Y$ be a function from X to Y, and let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous. Similarly $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous.

Proof

The function $\varphi\colon X \to Y$ is continuous if and only if

$$\{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$$

is open in X for all subsets V of Y that are open in Y (see Proposition 1.16). Let V be a subset of Y that is open in Y. Then $\Phi(\mathbf{x}) \subset V$ if and only if $\varphi(\mathbf{x}) \in V$. Also $\Phi(\mathbf{x}) \cap V \neq \emptyset$ if and only if $\varphi(\mathbf{x}) \in V$. The result therefore follows from the definitions of upper and lower hemicontinuity. ∎