MA3486—Fixed Point Theorems and Economic Equilibria
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1.3. The Extreme Value Theorem

We use the Bolzano-Weierstrass Theorem in order to prove the following important result.

Theorem 1.20 (The Multidimensional Extreme Value Theorem)

Let X be a closed bounded set in n-dimensional Euclidean space, and let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof

We first prove that if $f\colon X \to \mathbb{R}$ is a *bounded* continuous real-valued function on X then f attains a maximum and a minimum value on the set X. We then apply this result to show that all continuous real-valued functions on X are bounded. It will then follow that all continuous real-valued functions on X attain a maximum and a minimum value on the set X.

Thus suppose that $f\colon X \to \mathbb{R}$ is a bounded continuous real-valued function on the closed bounded set X. Then the set

$$\{f(\mathbf{x}) : \mathbf{x} \in X\}$$

of values of the function f is a bounded non-empty set and thus has a least upper bound M. There then exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X such that $f(\mathbf{x}_j) > M - 1/j$ for all positive integers j. The infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a bounded sequence, because it is contained in the bounded set X. It follows from the Bolzano-Weierstrass Theorem Theorem 1.4 that the infinite sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ that converges to some point $\mathbf{v}$ of $\mathbb{R}^m$. But then $\mathbf{v} \in X$, because the set X is closed (Lemma 1.10). But the continuity of f then ensures that $M = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = f(\mathbf{v})$. Therefore $f(\mathbf{x}) \leq f(\mathbf{v})$ for all points $\mathbf{x}$ of X.

Applying this result with f replaced by $-f$, we deduce also that there exists some point $\mathbf{u}$ of X with the property that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all points $\mathbf{x}$ of X. We have thus shown that if $f \colon X \to \mathbb{R}$ is both continuous and bounded, and if the set X is both closed and bounded, then there exist points $\mathbf{u}$ and $\mathbf{v} \in X$ such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Now let $f\colon X \to \mathbb{R}$ be any continuous real-valued function on X, and let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + f(\mathbf{x})^2}$$

for all $\mathbf{x} \in X$. Then the function g is both continuous on X, and $0 < g(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$. It follows from the result already obtained that there exists some point $\mathbf{w}$ of X such that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$. Moreover $g(\mathbf{w}) > 0$. Let K be a positive constant chosen large enough to ensure that $1/K^2 < g(\mathbf{w})$. Then $-K < f(\mathbf{x} < K$ for all points $\mathbf{x}$ of X. The function f is thus bounded in X. The general result therefore follows from the result already proved under the assumption that the function is both continuous and bounded. ∎

1.4. Lebesgue Numbers

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 1.21

Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof

Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then

$$B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for all positive integers j, the open ball

$$B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$.

The sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) that there would exist a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots.$$

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 1.10). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition

Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 1.21 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition

The *diameter* $\mathrm{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\mathrm{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\mathrm{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

Lemma 1.22

Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the $\mathrm{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof

Let b be a real number satisfying $0 < \sqrt{n}\, b < \delta$ and, for each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers, let $H_{(j_1, j_2, \ldots, j_n)}$ denote the hypercube in $\mathbb{R}^n$ defined such that

$$\begin{aligned} H_{(j_1,j_2,\ldots,j_n)} &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ &\qquad j_i b \leq x_i \leq (j_i + 1)b \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Note that if $\mathbf{u}$ and $\mathbf{v}$ are points of $H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers then $|u_i - v_i| < b$ for $i = 1, 2, \ldots, n$, and therefore $|\mathbf{u} - \mathbf{v}| \leq \sqrt{n}\, b < \delta$. Therefore the diameter of each hypercube $H_{(j_1,j_2,\ldots,j_n)}$ is less than δ.
The boundedness of the set X ensures that there are only finitely many n-tuples $(j_1, j_2, \ldots, j_n)$ of integers for which $X \cap H_{(j_1,j_2,\ldots,j_n)}$ is non-empty. It follows that X is covered by a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X, where each of these subsets is of the form $X \cap H_{(j_1,j_2,\ldots,j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers. These subsets all have diameter less than δ. The result follows. ∎

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition

A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 1.23

(The Multidimensional Heine-Borel Theorem) A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Proof

Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j} \right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 1.21 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 1.22 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\mathrm{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎