MA3486—Fixed Point Theorems and Economic Equilibria
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1. Review of Basic Results of Analysis in Euclidean Spaces

1.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be elements in $\mathbb{R}^n$, Let $p(t) = |t\mathbf{x} + \mathbf{y}|^2$ for all real numbers t. Then

$$\begin{aligned} p(t) &= (t\mathbf{x} + \mathbf{y}) \,.\, (t\mathbf{x} + \mathbf{y}) \\ &= t^2|\mathbf{x}|^2 + 2t\mathbf{x} \,.\, \mathbf{y} + |\mathbf{y}|^2 \end{aligned}$$

for all real numbers t. But $p(t) \geq 0$ for all real numbers t. It follows that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. This inqualty is known as *Schwarz's Inequality*.

Moreover, given any elements **x** and **y** of $\mathbf{R}^n$,

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

It follows that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. It follows from this inequality that

$$|\mathbf{x}-\mathbf{z}| \leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. This identity is known as the *Triangle Inequality*. It expresses the geometric result that the length of any side of a triangle in a Euclidean space of any dimension is the sum of the lengths of the other two sides of that triangle.

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.*

We refer to $\mathbf{p}$ as the *limit* $\lim_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 1.1

Let $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*. Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$*.*

A proof of Lemma 1.1 is to be found in Appendix A.

1.2. The Bolzano-Weierstrass Theorem

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.2

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

A proof of Theorem 1.2 is to be found in Appendix A.

Definition

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Theorem 1.3 (Bolzano-Weierstrass in One Dimension)

Every bounded sequence of real numbers has a convergent subsequence.

A proof of Theorem 1.3 is to be found in Appendix A.

Theorem 1.4 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

A proof of Theorem 1.4 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition

Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Lemma 1.5

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

A proof of Lemma 1.5 is to be found in Appendix A.

Proposition 1.6

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any finite collection of open sets in X is itself open in X.*

A proof of Proposition 1.6 is to be found in Appendix A.

Proposition 1.7

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

A proof of Proposition 1.7 is to be found in Appendix A.

Lemma 1.8

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.

A proof of Lemma 1.8 is to be found in Appendix A.

Definition

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Proposition 1.9

Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any finite collection of closed sets in X is itself closed in X.*

A proof of Proposition 1.9 is to be found in Appendix A.

Lemma 1.10

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

A proof of Lemma 1.10 is to be found in Appendix A.

Definition

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.*

The function $f\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 1.11

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

A proof of Lemma 1.11 is to be found in Appendix A.

Lemma 1.12

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

A proof of Lemma 1.12 is to be found in Appendix A.

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 1.13

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f \colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

A proof of Proposition 1.13 is to be found in Appendix A.

Proposition 1.14

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

A proof of Proposition 1.14 is to be found in Appendix A.

Lemma 1.15

Let X be a subset of $\mathbb{R}^m$, let $f \colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f| \colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

A proof of Proposition 1.15 is to be found in Appendix A.

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 1.16

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

A proof of Proposition 1.16 is to be found in Appendix A.

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 1.17

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.

Proof

Let F be a subset of Y that is closed in Y, and let let $V = Y \setminus F$. Then V is open in Y. It follows from Proposition 1.16 that $\varphi^{-1}(V)$ is open in X. But

$$\varphi^{-1}(V) = \varphi^{-1}(Y \setminus F) = X \setminus \varphi^{-1}(F).$$

Indeed let $\mathbf{x} \in X$. Then

$$\begin{aligned} & \mathbf{x} \in \varphi^{-1}(V) \\ \iff & \mathbf{x} \in \varphi^{-1}(Y \setminus F) \\ \iff & \varphi(\mathbf{x}) \in Y \setminus F \\ \iff & \varphi(\mathbf{x}) \notin F \\ \iff & \mathbf{x} \notin \varphi^{-1}(F) \\ \iff & \mathbf{x} \in X \setminus \varphi^{-1}(F). \end{aligned}$$

It follows that the complement $X \setminus \varphi^{-1}(F)$ of $\varphi^{-1}(F)$ in X is open in X, and therefore $\varphi^{-1}(F)$ itself is closed in X, as required. ∎

Lemma 1.18

Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.

Proof

Let F be a subset of X. Then F is closed in X if and only if, given any point $\mathbf{p}$ of X for which $\mathbf{p} \notin F$, there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. It follows easily from this that is F is closed in $\mathbb{R}^n$ then F is closed in X.

Conversely suppose that F is closed in X, where X itself is closed in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that satisfies $\mathbf{p} \notin F$. Then either $\mathbf{p} \in X$ or $\mathbf{p} \notin X$.

Suppose that $\mathbf{p} \in X$. Then there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ.

Otherwise $\mathbf{p} \notin X$. Then there exists some strictly positive real number δ such that there is no point of X whose distance from the point $\mathbf{p}$ is less than δ, because X is closed in $\mathbb{R}^n$. But $F \subset X$. It follows that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. We conclude that the set F is closed in $\mathbb{R}^n$, as required. ∎

The following result, together with its generalizations, is sometimes referred to as the *Glueing Lemma*.

Lemma 1.19 (Glueing Lemma)

Let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping a subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$. Let $F_1, F_2, \ldots, F_k$ be a finite collection of subsets of X such that F_i is closed in X for $i = 1, 2, \ldots, k$ and

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ is continuous on X if and only if the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Proof

Suppose that $\varphi\colon X \to \mathbb{R}^n$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi \colon X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi\colon X \to \mathbb{R}^n$ is continuous, as required. ∎