MA3486—Fixed Point Theorems and Economic Equilibria
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Hilary Term 2018
Appendix A

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A. Proofs of Basic Results of Real Analysis

Lemma 1.1

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.

Proof of Lemma 1.1

Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

The real number system satisfies the *Least Upper Bound Principle*:

> *Any set of real numbers which is non-empty and bounded above has a least upper bound.*

Let S be a set of real numbers which is non-empty and bounded above. The least upper bound, or *supremum*, of the set S is denoted by $\sup S$, and is characterized by the following two properties:

(i) $x \leq \sup S$ for all $x \in S$;

(ii) if u is a real number, and if $x \leq u$ for all $x \in S$, then $\sup S \leq u$.

A straightforward application of the Least Upper Bound guarantees that any set of real numbers that is non-empty and bounded below has a greatest lower bound, or *infimum*. The greatest lower bound of such a set S of real numbers is denoted by $\inf S$.

Theorem 1.2

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof of Theorem 1.2

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.
If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Theorem 1.3

Every bounded sequence of real numbers has a convergent subsequence.

Proof of Theorem 1.3

Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.2 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.2. This completes the proof of the Bolzano-Weierstrass Theorem. $\blacksquare$

We introduce some terminology and notation for discussing convergence along subsequences of bounded sequences of points in Euclidean spaces. This will be useful in proving the multi-dimensional version of the Bolzano-Weierstrass Theorem.

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$, let J be an infinite subset of the set $\mathbb{N}$ of positive integers, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{p}$ is the *limit* of $\mathbf{x}_j$ as j *tends to infinity in the set* J, and write "$\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ in J" if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \in J$ and $j \geq N$.*

The one-dimensional version of the Bolzano-Weierstrass Theorem asserts that every bounded sequence of real numbers has a convergent subsequence. We seek to generalize this result to bounded sequences of points in n-dimensional Euclidean space $\mathbb{R}^n$. Now the one-dimensional version of the Bolzano-Weierstrass Theorem is equivalent to the following statement:

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, there exists an infinite subset J of the set $\mathbb{N}$ of positive integers and a real number p such that $x_j \to p$ as $j \to +\infty$ in J.*

Given an infinite subset J of $\mathbb{N}$, the elements of J can be labelled as $k_1, k_2, k_3, \ldots$, where $k_1 < k_2 < k_3 < \cdots$, so that k_1 is the smallest positive integer belonging of J, k_2 is the next smallest, etc. Therefore any standard result concerning convergence of sequences of points can be applied in the context of the convergence of subsequences of a given sequence of points. The following result is therefore a direct consequence of the one-dimensional Bolzano-Weierstrass Theorem:

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given an infinite subset J of the set $\mathbb{N}$ of positive integers, there exists an infinite subset K of J and a real number p such that $x_j \to p$ as $j \to +\infty$ in K.*

The above statement in fact corresponds to the following assertion:—

Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ *of real numbers, and given any subsequence*

$$x_{k_1}, x_{k_2}, x_{k_3}, \cdots$$

of the given infinite sequence, there exists a convergent subsequence

$$x_{k_{m_1}}, x_{k_{m_2}}, x_{k_{m_3}}, \ldots$$

of the given subsequence. Moreover this convergent subsequence of the given subsequence is itself a convergent subsequence of the given infinite sequence, and it contains only members of the given subsequence of the given sequence.

The basic principle can be presented purely in words as follows:

> *Given a bounded sequence of real numbers, and given a subsequence of that original given sequence, there exists a convergent subsequence of the given subsequence. Moreover this subsequence of the subsequence is a convergent subsequence of the original given sequence.*

We employ this principle in the following proof of the Multidimensional Bolzano-Weierstrass Theorem.

Theorem 1.4

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

Proof of Theorem 1.4

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$, and, for each positive integer j, and for each integer i between 1 and n, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then

$$\mathbf{x}_j = \Big((\mathbf{x}_j)_1, (\mathbf{x}_j)_2, \ldots, (\mathbf{x}_j)_n\Big).$$

for all positive integers j. It follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_1 of the set $\mathbb{N}$ of positive integers and a real number p_1 such that $(\mathbf{x}_j)_1 \to p_1$ as $j \to +\infty$ in J_1.

Let k be an integer between 1 and $n-1$. Suppose that there exists an infinite subset J_k of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_k$ such that, for each integer i between 1 and k, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_k. It then follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_{k+1} of J_k and a real number p_{k+1}, such that $(\mathbf{x}_j)_{k+1} \to p_{k+1}$ as $j \to +\infty$ in J_{k+1}. Moreover the requirement that $J_{k+1} \subset J_k$ then ensures that, for each integer i between 1 and $k+1$, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_{k+1}. Repeated application of this result then ensures the existence of an infinite subset J_n of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_n$ such that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_n. Let

$$J_n = \{k_1, k_2, k_3, \ldots\},$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\lim\limits_{j\to+\infty} (\mathbf{x}_{k_j})_i = p_i$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 1.1 that $\lim\limits_{j\to+\infty} \mathbf{x}_{k_j} = \mathbf{p}$. The result follows. ∎

Lemma 1.5

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

Proof of Lemma 1.5

Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Proposition 1.6

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any *finite* collection of open sets in X is itself open in X.

Proof of Proposition 1.6

The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). $\blacksquare$

Proposition 1.7

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

Proof of Proposition 1.7

First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$. Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 1.5), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 1.6). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

Lemma 1.8

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.

Proof of Lemma 1.8

Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 1.5. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. $\blacksquare$

Lemma 1.10

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

Proof of Lemma 1.10

The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 1.8 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

Lemma 1.11

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof of Lemma 1.11

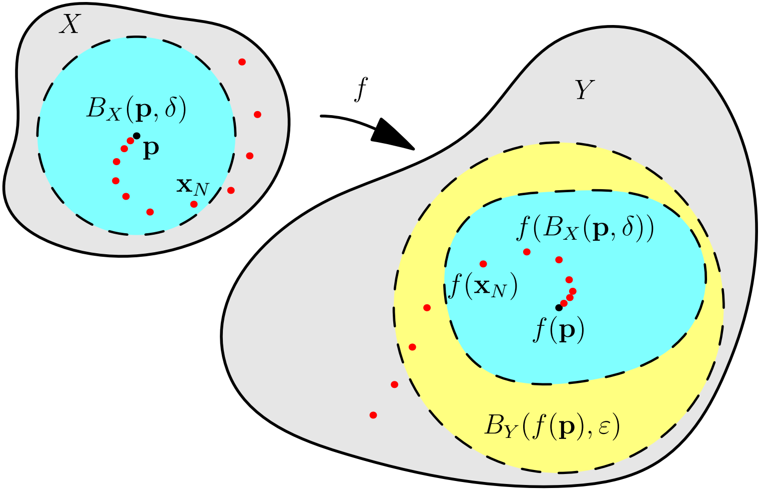
Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. $\blacksquare$

Lemma 1.12

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

Proof of Lemma 1.12

Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$.


X
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
$\mathbf{x}_N$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{x}_N)$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Proposition 1.11

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof of Proposition 1.11

Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 1.11. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Proposition 1.14

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

Proof of Proposition 1.14

First we prove that $f + g$ is continuous. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(\mathbf{x}) - f(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|(f+g)(\mathbf{x})-(f+g)(\mathbf{p})| \leq |f(\mathbf{x})-f(\mathbf{p})|+|g(\mathbf{x})-g(\mathbf{p})| < \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus the function $f + g$ is continuous at $\mathbf{p}$.

The function $-g$ is also continuous at $\mathbf{p}$, and $f - g = f + (-g)$. It follows that the function $f - g$ is continuous at $\mathbf{p}$.

Next we prove that $f \cdot g$ is continuous. Let some strictly positive real number ε be given. There exists some strictly positive real number δ_0 such that $|f(\mathbf{x}) - f(\mathbf{p})| < 1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < 1$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Let M be the maximum of $|f(\mathbf{p})| + 1$ and $|g(\mathbf{p})| + 1$. Then $|f(\mathbf{x})| < M$ and $|g(\mathbf{x})| < M$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Now

$$f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p}) = (f(\mathbf{x}) - f(\mathbf{p}))g(\mathbf{x}) + f(\mathbf{p})(g(\mathbf{x}) - g(\mathbf{p})),$$

and thus

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| \leq M\Big(|f(\mathbf{x}) - f(\mathbf{p})| + |g(\mathbf{x}) - g(\mathbf{p})|\Big)$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$.

There then exists some strictly positive real number δ, where $0 < \delta \leq \delta_0$, such that

$$|f(\mathbf{x}) - f(\mathbf{p})| < \frac{\varepsilon}{2M} \quad \text{and} \quad |g(\mathbf{x}) - g(\mathbf{p})| < \frac{\varepsilon}{2M}$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| < \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f \cdot g$ is continuous at $\mathbf{p}$.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. $\blacksquare$

Lemma 1.15

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

Proof of Lemma 1.15

Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big||f(\mathbf{x})| - |f(\mathbf{p})|\Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

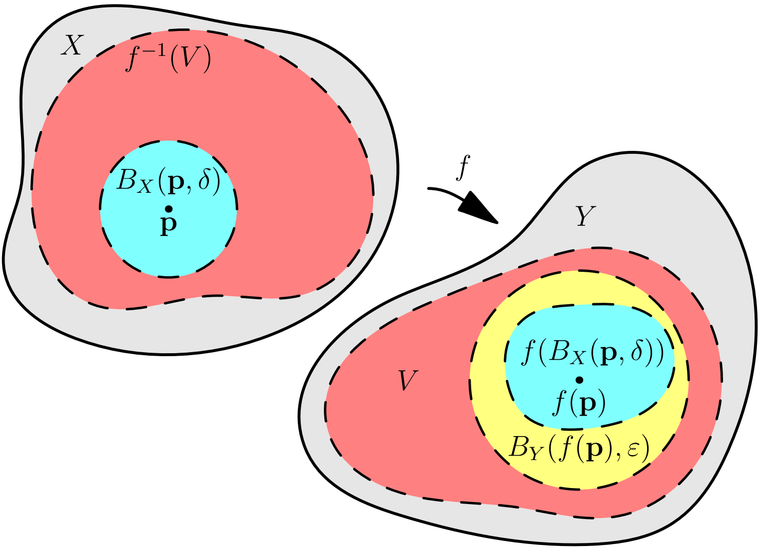
for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

Proposition 1.16

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

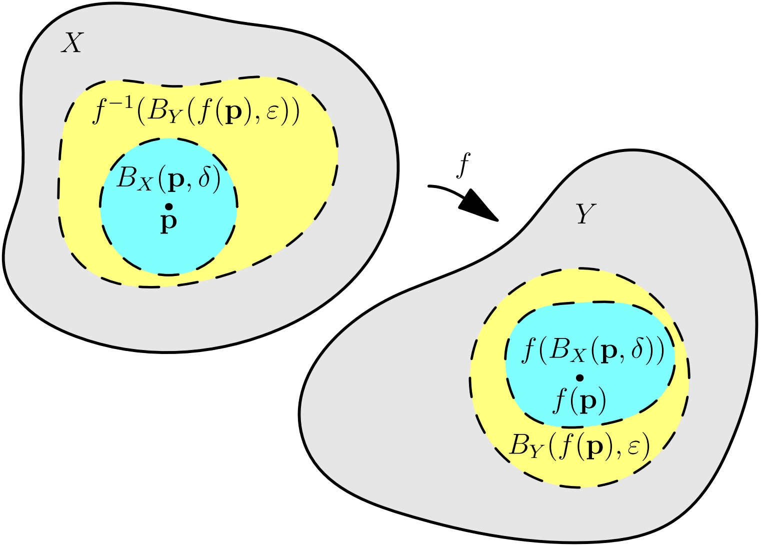
Proof of Proposition 1.16

Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.



X
$f^{-1}(V)$
$B_X(\mathbf{p},\delta)$
$\mathbf{p}$
f
Y
V
$f(B_X(\mathbf{p},\delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}),\varepsilon)$

Conversely suppose that $f : X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$.


X
$f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 1.5, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎