MA3486 Fixed Point Theorems and Economic Equilibria
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8.3. Zero-Sum Two-Person Games

Example

Consider the following hand game. This is a zero-sum two-person game. At each go, the two players present simultaneously either and open hand or a fist. If both players present fists, or if both players present open hands, then no money changes hands. If one player presents a fist and the other player presents an open hand then the player presenting the fist receives ten cents from the player presenting the open hand.

The payoff for the first player can be represented by the following payoff matrix:

$$\begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}.$$

In this matrix the entry in the first row represent the payoffs when the first player presents an open hand; those in the second row represent the payoffs when the first player presents a fist. The entries in the first column represent the payoff when the second player presents an open hand; those in the second column represent the payoffs when the second player presents a fist. In this game the second player, choosing the best strategy, is always going to play a fist, because that reduces the payoff for the first player, whatever the first player chooses to play. Similarly the first player, choosing the best strategy, is going to play a fist, because that maximizes the payoff for the first player whatever the second player does. Thus in this game, both players choosing the best strategies, play fists.

It should be noticed that, in this situation, if the second player always plays a fist, the first player would not be tempted to move from a strategy of always playing a fist in order get a better payoff. Similarly if the first player always plays a fist, then the second player would not be tempted to move from a strategy of always playing a fist in order to reduce the payoff to the first player. This is a very simple example of a *Nash Equilibrium*. This equilibrium arises because the element in the second row and second column of the payoff matrix is simultaneously the largest element in its column and the smallest element in its row. Matrix elements with this property as said to be *saddle points* of the matrix.

Example

Now consider the game of *Rock, Paper, Scissors*. This game has a long history, and versions of this game were well-established in China and Japan in particular for many centuries.

Two players simultaneously present hand symbols representing *Rock* (a closed fist), *Paper* (a flat hand), or *Scissors* (first two fingers outstretched in a 'V'). Paper beats Rock, Scissors beats Paper, Rock beats Scissors. If both players present the same hand symbol then that round is a draw.

Ordering the strategies for the playes in the order *Rock* (1st), *Paper* (2nd) and *Scissors* (3rd), the payoff matrix for the first player is the following:—

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The entry in the ith row and jth column of this payoff matrix represents the return to the first player on a round of the game if the first player plays strategy i and the second player plays strategy j.

A *pure strategy* would be one in which a player presents the same hand symbol in every round. But it is not profitable for any player in this game to adopt a pure strategy. If the first player adopts a strategy of playing *Paper*, then the second player, on observing this, would adopt a strategy of always playing *Scissors*, and would beat the first player on every round. A preferable strategy, for each player, is the *mixed strategy* of playing *Rock*, *Paper* and *Scissors* with equal probability, and seeking to ensure that the sequence of plays is as random as possible.

Let us denote by M the payoff matrix above. A *mixed strategy* for the first player is one in which, on any given round *Rock* is played with probability p_1, *Paper* is played with probability p_2 and *Scissors* is played with probability p_3. The mixed strategies for the first player can therefore be represented by points of a triangle Δ_P, where

$$\Delta_P = \left\{ (p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 \geq 0, \;\; p_2 \geq 0, \;\; p_3 \geq 0, \;\; p_1 + p_2 + p_3 = 1 \right\}.$$

A mixed strategy for the second player is one in which *Rock* is played with probability q_1, *Paper* with probability q_2 and *Scissors* with probability q_3. The mixed strategies for the second player can therefore be represented by points of a triangle Δ_Q, where

$$\Delta_Q = \{ (q_1, q_2, q_3) \in \mathbb{R}^m : q_1 \geq 0, \;\; q_2 \geq 0, \;\; q_3 \geq 0, \;\; q_1 + q_2 + q_3 = 1 \}.$$

Let $\mathbf{p} \in \Delta_P$ represent the mixed strategy chosen by the first player, and $\mathbf{q} in \Delta_Q$ the mixed strategy chosen by the second player, where

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{q} = (q_1, q_2, q_3).$$

Let M_{ij} the payoff for the first player when the first player plays strategy i and the second player plays strategy j. Then M_{ij} is the entry in the ith row and jth column of the payoff matrix M. In matrix equations we consider $\mathbf{p}$ and $\mathbf{q}$ to be column vectors, denoting their transposes by the row matrices $\mathbf{p}^T$ and $\mathbf{q}^T$. The *expected payoff* for the first player is then $f(\mathbf{p}, \mathbf{q})$, where

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{3} \sum_{j=1}^{3} p_i M_{ij} q_j.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, where

$$p_1^* = p_2^* = p_3^* = \tfrac{1}{3}.$$

Then $\mathbf{p}^{*T} M = (0, 0, 0)$, and therefore

$$f(\mathbf{p}^*, \mathbf{q}) = 0$$

for all $\mathbf{q} \in \Delta_Q$. Similarly let $\mathbf{q}^* = (q_1^*, q_2^*, q_3^*)$, where

$$q_1^* = q_2^* = q_3^* = \tfrac{1}{3}.$$

Then

$$f(\mathbf{p}, \mathbf{q}^*) = 0$$

for all $\mathbf{p} \in \Delta_Q$. Thus the inequalities

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

are satisfied for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_q$, because each of the quantities occurring is equal to zero.

Were the first player to adopt a mixed strategy $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, p_3)$, $p_i \geq 0$ for $i = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$, the second player could adopt mixed strategy $\mathbf{q}$, where $\mathbf{q} = (q_1, q_2, q_3) = (p_3, p_1, p_2)$. The payoff $f(\mathbf{p}, \mathbf{q})$ is then

$$\begin{aligned} f(\mathbf{p}, \mathbf{q}) &= -p_1 q_2 + p_1 q_3 - p_2 q_3 + p_2 q_1 - p_3 q_1 + p_3 q_2 \\ &= -p_1^2 + p_1 p_2 - p_2^2 + p_2 p_3 - p_3^2 + p_3 p_1 \\ &= -\tfrac{1}{6}\Big((2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 \\ &\qquad + (2p_3 - p_1 - p_2)^2\Big) \\ &\leq 0. \end{aligned}$$

Moreover if $f(\mathbf{p}, \mathbf{q}) = 0$, where $q_1 = p_3$, $q_2 = p_1$ and $q_3 = p_2$, then

$$(2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 + (2p_3 - p_1 - p_2)^2 = 0$$

and therefore $2p_1 = p_2 + p_3$, $2p_2 = p_3 + p_1$ and $2p_3 = p_1 + p_2$. But then

$$3p_1 = 3p_2 = 3p_3 = p_1 + p_2 + p_3 = 1,$$

and thus $\mathbf{p} = \mathbf{p}^*$. It follows that if $\mathbf{p} \in \Delta_Q$ and $\mathbf{p} \neq \mathbf{p}^*$ then there exists $\mathbf{q} \in \Delta_Q$ for which $f(\mathbf{p}, \mathbf{q}) < 0$. Thus if the first player adopts a mixed strategy other than the strategy $\mathbf{p}^*$ in which *Rock*, *Paper*, *Scissors* are played with equal probability on each round, there is a mixed strategy for the second player that ensures that the average payoff for the first player is negative, and thus the first player will lose in the long run over many rounds. Thus strategy $\mathbf{p}^*$ is the only sensible mixed strategy that the first player can adopt. The corresponding strategy $\mathbf{q}^*$ is the only sensible mixed strategy that the second player can adopt. The average payoff for each player is then equal to zero.

8.4. Von Neumann's Minimax Theorem

In 1920, John Von Neumann published a paper entitled "Zur Theorie der Gesellschaftsspielle" (*Mathematische Annalen*, Vol. 100 (1928), pp. 295–320). The title translates as "On the Theory of Social Games". This paper included a proof of the following "Minimax Theorem", which made use of the Brouwer Fixed Point Theorem. An alternative proof using results concerning convexity was presented in the book *On the Theory of Games and Economic Behaviour* by John Von Neumann and Oskar Morgenstern (Princeton University Press, 1944). George Dantzig, in a paper published in 1951, showed how the theorem could be solved using linear programming methods (see Joel N. Franklin, *Methods of Mathematical Economics*, (Springer Verlag, 1980, republished by SIAM in 1982).

Theorem 8.7 (Von Neumann's Minimax Theorem)

Let M be an $m \times n$ matrix, and let

$$\begin{aligned} \Delta_P &= \left\{ (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geq 0 \text{ for } i = 1, 2, \ldots, m, \text{ and } \sum_{i=1}^{m} p_i = 1 \right\}, \\ \Delta_Q &= \left\{ (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{j=1}^{n} q_j = 1 \right\}, \end{aligned}$$

and let

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j} p_i q_j$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Then there exist $\mathbf{p}^* \in \Delta_P$ and $\mathbf{q}^* \in \Delta_Q$ such that

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$.

Proof

Let $f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{p} \in \Delta_P$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p}, \mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{q})\}.$$

An application of Berge's Maximum Theorem (Theorem 4.24) ensures that the functions $\mu_P\colon \Delta_P \to \mathbb{R}$ and $\mu_Q\colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P\colon \Delta_Q \rightrightarrows \Delta_P$ and $Q\colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs (see Proposition 4.11). Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi\colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem (Theorem 8.6) then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required. ∎

9.1. The Existence of Equilibria in an Exchange Economy

We consider an exchange economy consisting of a finite number of commodities and a finite number of households, each provided with an initial endowment of each of the commodities. The commodities are required to be *infinitely divisible*: this means that a household can hold an amount x of that commodity for any non-negative real number x. (Thus salt, for example, could be regarded as an 'infinitely divisible' quantity whereas cars cannot: it makes little sense to talk about a particular household owning 2.637 of a car, for example, though such a household may well own 2.637 kilograms of salt.) Now the households may well wish to exchange commodities with one another so as improve on their initial endowment. They might for example seek to barter commodities with one another: however this method of redistribution would not work very efficiently in a large economy.

Alternatively they might attempt to set up a price mechanism to simplify the task of redistributing the commodities. Thus suppose that each commodity is assigned a given price. Then each household could sell its initial endowment to the market, receiving in return the value of its initial endowment at the given prices. The household could then purchase from the market a quantity of each commodity so as to maximize its own preference, subject to the constraint that the total value of the commodities purchased by any household cannot exceed the value of its initial endowment at the given prices. The problem of redistribution then becomes one of fixing prices so that there is exactly enough of each commodity to go around: if the price of any commodity is too low then the demand for that commodity is likely to outstrip supply, whereas if the price is too high then supply will exceed demand. A *Walras equilibrium* is achieved if prices can be found so that the supply of each commodity matches its demand. We shall use the *Brouwer fixed point theorem* to prove the existence of a Walras equilibrium in this idealized economy.

Let our exchange economy consist of n commodities and m households. We suppose that household h is provided with an initial endowment $\overline{x}_{hi}$ of commodity i, where $\overline{x}_{hi} \geq 0$. Thus the initial endowment of household h can be represented by a vector $\overline{\mathbf{x}}_h$ in $\mathbb{R}^n$ whose ith component is $\overline{x}_{hi}$. The prices of the commodities are given by a price vector $\mathbf{p}$ whose ith component p_i specifies the price of a unit of the ith commodity: a price vector $\mathbf{p}$ is required to satisfy $p_i \geq 0$ for all i. Then the value of the initial endowment of household h at the given prices is $\mathbf{p}.\overline{\mathbf{x}}_h$. Let $x_{hi}(\mathbf{p})$ be the quantity of commodity i that household h seeks to purchase at prices $\mathbf{p}$, and let $\mathbf{x}_h(\mathbf{p}) \in \mathbb{R}^n$ be the vector whose ith component is $x_{hi}(\mathbf{p})$. The budget constraint certainly ensures that $\mathbf{p}.(\mathbf{x}_h(\mathbf{p}) - \overline{\mathbf{x}}_h) \leq 0$ (i.e., the value of the goods purchased cannot exceed the value of the initial endowment at the given prices).

We assume that the value of the commodities that each household seeks to purchase is equal to the value of its initial endowment, and thus $\mathbf{p}.\mathbf{x}_h(\mathbf{p}) = \mathbf{p}.\overline{\mathbf{x}}_h$. Also the preferences of the household will only depend on the relative prices of the commodities, and therefore $\mathbf{x}_h(\lambda\mathbf{p}) = \mathbf{x}_h(\mathbf{p})$ for all $\lambda > 0$.

Now the total supply of each commodity in the economy is represented by the vector $\sum_h \overline{\mathbf{x}}_h$, and the total demand at prices $\mathbf{p}$ is represented by $\sum_h \mathbf{x}_h(\mathbf{p})$. The *excess demand* in the economy at prices $\mathbf{p}$ is therefore represented by the vector $\mathbf{z}(\mathbf{p})$, where $\mathbf{z}(\mathbf{p}) = \sum_h (\mathbf{x}_h(\mathbf{p}) - \overline{\mathbf{x}}_h)$. Let $z_i(\mathbf{p})$ be the ith component of $\mathbf{z}(\mathbf{p})$. Then $z_i(\mathbf{p}) > 0$ when the demand for the ith commodity exceeds supply, whereas $z_i(\mathbf{p}) < 0$ when the supply exceeds demand. Note that $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$ for any price vector $\mathbf{p}$. This identity, known as *Walras' Law*, follows immediately on summing the budget constraint $\mathbf{p}.\mathbf{x}_h(\mathbf{p}) = \mathbf{p}.\overline{\mathbf{x}}_h$ over all households.

Consider an exchange economy consisting of a finite number of infinitely divisible commodities and a finite number of households. Let the excess demand in the economy at prices $\mathbf{p}$ be given by $\mathbf{z}(\mathbf{p})$, where

(i) the excess demand vector $\mathbf{z}(\mathbf{p})$ is well-defined for any price vector $\mathbf{p}$, and depends continuously on $\mathbf{p}$,

(ii) $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$ for any price vector $\mathbf{p}$ (Walras' Law).

It then follows from Corollary 7.4 that there exist equilibrium prices $\mathbf{p}^*$ at which $z_i(\mathbf{p}^*) \leq 0$ for all i.

The proof of the existence of Walras equilibria can readily be generalized to *Arrow-Debreu* models where economic activity is carried out by both households and firms. The problem of existence of equilibria was studied by L. Walras in the 1870s, though a rigorous proof of the existence of equilibria was not found till the 1930s, when A. Wald proved existence for a limited range of economic models. Proofs of existence using the Brouwer Fixed Point Theorem, or a more general fixed point theorem due to Katukani, were first published in 1954 by K. J. Arrow and G. Debreu and by L. McKenzie. Subsequent research has centred on problems of uniqueness and stability, and the existence theorems have been generalized to economies with an infinite number of commodities and economic agents (households and firms). An alternative approach to the existence theorems using techniques of differential topology was pioneered by G. Debreu and by S. Smale.

More detailed accounts of the theory of ‘general equilibrium’ can be found in, for example, *The theory of value*, by G. Debreu, *General competitive analysis*, by K. J. Arrow and F. H. Hahn, or *Economics for mathematicians* by J. W. S. Cassels.