MA3486 Fixed Point Theorems and Economic Equilibria
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8. Convexity and the Kakutani Fixed Point Theorem

8.1. Convex Subsets of Euclidean Spaces

Definition

A subset X of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{u} + t\mathbf{v} \in X$ for all points $\mathbf{u}$ and $\mathbf{v}$ of X and for all real numbers t satisfying $0 \leq t \leq 1$.

Lemma 8.1

An simplex in a Euclidean space is a convex subset of that Euclidean space.

Proof

Let σ be a q-simplex in n-dimensional Euclidean space, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of σ. Let $\mathbf{u}$ and $\mathbf{v}$ be points of σ. Then there exist non-negative real numbers $y_0, y_1, \ldots, y_q$ and $z_0, z_1, \ldots, z_q$, where $\sum_{i=0}^{q} y_i = 1$ and $\sum_{i=0}^{q} z_i = 1$, such that

$$\mathbf{u} = \sum_{i=0}^{q} y_i \mathbf{w}_i, \quad \mathbf{v} = \sum_{i=0}^{q} z_i \mathbf{w}_i.$$

Then

$$(1-t)\mathbf{u} + t\mathbf{v} = \sum_{i=0}^{q} ((1-t)y_i + tz_i)\mathbf{w}_i.$$

Moreover $(1-t)y_i + tz_i \geq 0$ for $i = 0, 1, \ldots, q$ and for all real numbers t satisfying $0 \leq t \leq 1$. Also

$$\sum_{i=0}^{q}((1-t)y_i + tz_i) = (1-t)\sum_{i=0}^{q} y_i + t\sum_{i=0}^{q} z_i = 1.$$

It follows that $(1-t)\mathbf{u} + t\mathbf{v} \in \sigma$. Thus σ is a convex subset of $\mathbb{R}^n$. ∎

Lemma 8.2

Let X be a convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let σ be a simplex contained in $\mathbb{R}^n$. Suppose that the vertices of σ belong to X. Then $\sigma \subset X$.

Proof

We prove the result by induction on the dimension q of the simplex σ. The result is clearly true when $q = 0$, because in that case the simplex σ consists of a single point which is the unique vertex of the simplex. Thus let σ be a q-dimensional simplex, and suppose that the result is true for all $(q-1)$-dimensional simplices whose vertices belong to the convex set X. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of σ. Let $\mathbf{x}$ be a point of σ. Then there exist non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$ such that $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{w}_i$. If $t_0 = 1$ then $\mathbf{x} = \mathbf{w}_0$, and therefore $\mathbf{x} \in X$.

It remains to consider the case when $t_0 < 1$. In that case let $s_i = t_i/(1 - t_0)$ for $i = 1, 2, \ldots, q$, and let

$$\mathbf{v} = \sum_{i=1}^{q} s_i \mathbf{w}_i.$$

Now $s_i \geq 0$ for $i = 1, 2, \ldots, q$, and

$$\sum_{i=1}^{q} s_i = \frac{1}{1 - t_0} \sum_{i=1}^{q} t_i = \frac{1}{1 - t_0} \left(\sum_{i=0}^{q} t_i - t_0 \right) = 1,$$

It follows that $\mathbf{v}$ belongs to the proper face of σ spanned by vertices $\mathbf{w}_1, \ldots, \mathbf{w}_q$. The induction hypothesis then ensures that $\mathbf{v} \in X$. But then

$$\mathbf{x} = t_0 \mathbf{w}_0 + (1 - t_0)\mathbf{v},$$

where $\mathbf{w}_0 \in X$ and $\mathbf{v} \in X$ and $0 \leq t_0 \leq 1$. It follows from the convexity of X that $\mathbf{x} \in X$, as required. ∎

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^{\ltimes}$. A point $\mathbf{x}$ of X is said to belong to the *topological interior* of X if there exists some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset X$, where

$$B(\mathbf{x}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{x}| < \delta\}.$$

Lemma 8.3

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X$ and $0 < t < 1$. Suppose that either $\mathbf{u}$ or $\mathbf{v}$ belongs to the topological interior of X. Then $\mathbf{x}$ belongs to the topological interior of X.

Proof

Suppose that $\mathbf{v}$ belongs to the topological interior of X. Then there exists $\delta > 0$ such that $B(\mathbf{v}, \delta) \subset X$, where

$$B(\mathbf{v}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{v}| < \delta\}.$$

We claim that $B(\mathbf{x}, t\delta) \subset X$.
Let $\mathbf{x}' \in B(\mathbf{x}, t\delta)$, and let

$$\mathbf{z} = \frac{1}{t}(\mathbf{x}' - \mathbf{x}).$$

Then $\mathbf{v} + \mathbf{z} \in B(\mathbf{v}, \delta)$ and

$$\mathbf{x}' = (1 - t)\mathbf{u} + t(\mathbf{v} + \mathbf{z}),$$

and therefore $\mathbf{x}' \in X$. This proves the result when $\mathbf{v}$ belongs to the topological interior of X. The result when $\mathbf{u}$ belongs to the topological interior of X then follows on interchanging $\mathbf{u}$ and $\mathbf{v}$ and replacing t by $1 - t$. The result follows. ∎

Proposition 8.4

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$ whose topological interior contains the origin, let S^{n-1} be the unit sphere in $\mathbb{R}^n$, defined such that

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous.

Proof

Let $\mathbf{u}_0 \in S^{n-1}$, let $t_0 = \lambda(\mathbf{u}_0)$, and let some positive real number ε be given, where $0 < \varepsilon < t_0$. It follows from Lemma 8.3 that $(t_0 - \varepsilon)\mathbf{u}$ belongs to the topological interior of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 - \varepsilon)\mathbf{u}$ that there exists some positive real number δ_1 such that $(t_0 - \varepsilon)\mathbf{u} \in X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_1$. Therefore $\lambda(\mathbf{u}) \geq t_0 - \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_1$.

Next we note that $(t_0 + \varepsilon)\mathbf{u}_0 \notin X$. Now X is closed in $\mathbb{R}^n$, and therefore the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$ is open. It follows that there exists an open ball of positive radius about the point $(t_0 + \varepsilon)\mathbf{u}_0$ that is wholly contained in the complement of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 + \varepsilon)\mathbf{u}$ that there exists some positive real number δ_2 such that $(t_0 + \varepsilon)\mathbf{u} \notin X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. It then follows from the convexity of X that $t\mathbf{u} \notin X$ for all positive real numbers t satisfying $t \geq t_0 + \varepsilon$. Therefore $\lambda(\mathbf{u}) \leq t_0 + \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$\lambda(\mathbf{u}_0) - \varepsilon \leq \lambda(\mathbf{u}) \leq \lambda(\mathbf{u}_0) + \varepsilon$$

for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta$. The result follows. ∎

Proposition 8.5

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then there exists a continuous map $r\colon \mathbb{R}^n \to X$ such that $r(\mathbb{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

Proof

We first prove the result in the special case in which the convex set X has non-empty topological interior. Without loss of generality, we may assume that the origin of $\mathbb{R}^n$ belongs to the topological interior of X. Let

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous (Proposition 8.4).

We may therefore define a function $r\colon \mathbb{R}^n \to X$ such that

$$r(\mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in X; \\ |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x} & \text{if } \mathbf{x} \not\in X. \end{cases}$$

Let $\mathbf{x} \in X$ and let $\mathbf{u} = |\mathbf{x}|^{-1}\mathbf{x}$. Then $\mathbf{x} = |\mathbf{x}|\,\mathbf{u}$, $|\mathbf{x}| \leq \lambda(\mathbf{u})$ and $\lambda(\mathbf{u})\mathbf{u} \in X$. It follows from Lemma 8.3 that if $|\mathbf{x}| < \lambda(\mathbf{u})$ then the point $\mathbf{x}$ belongs to the topological interior of $\mathbf{u}$. Thus if the point $\mathbf{x}$ of X belongs to the closure of the complement $\mathbb{R}^n \setminus X$ of X then it does not belong to the topological interior of X, and therefore $|\mathbf{x}| = \lambda(|\mathbf{x}|^{-1}\mathbf{x})$, and therefore

$$\mathbf{x} = |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x}.$$

The function r defined above is therefore continuous on the closure of $\mathbb{R}^n \setminus X$. It is obviously continuous on X itself. It follows that $r\colon \mathbb{R}^n \to X$ is continuous. This proves the result in the case when the topological interior of the set X is non-empty.

We now extend the result to the case where the topological interior of X is empty. Now the number of points in an affinely independent list of points of $\mathbb{R}^n$ cannot exceed $n+1$. It follows that there exists an integer q not exceeding n such that the convex set X contains a $q+1$ affinely independent points but does not contain $q+1$ affinely independent points. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be affinely independent points of X. Let V be the q-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors

$$\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0, \ldots, \mathbf{w}_q - \mathbf{w}_0.$$

Now if there were to exist a point $\mathbf{x}$ of X for which $\mathbf{x} - \mathbf{w}_0 \notin V$ then the points $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q, \mathbf{x}$ would be affinely independent. The definition of q ensures that this is not the case. Thus if

$$X_V = \{\mathbf{x} - \mathbf{w}_0 : \mathbf{x} \in X\}.$$

then $X_V \subset V$. Moreover X_V is a closed convex subset of V.

Now it follows from Lemma 8.2 that the convex set X_V contains the q-simplex with vertices

$$\mathbf{0},\ \mathbf{w}_1 - \mathbf{w}_0,\ \mathbf{w}_2 - \mathbf{w}_0, \ldots\ \mathbf{w}_q - \mathbf{w}_0.$$

This q-simplex has non-empty topological interior with respect to the vector space V. It follows that X_V has non-empty topological interior with respect to V. It therefore follows from the result already proved that there exists a continuous function $r_V \colon V \to X_V$ that satisfies $r_V(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X_V$. Basic linear algebra ensures the existence of a linear transformation $T \colon \mathbb{R}^n \to V$ satisfying $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. Let

$$r(\mathbf{x}) = r_V(T(\mathbf{x} - \mathbf{w}_0)) + \mathbf{w}_0$$

for all $\mathbf{x} \in \mathbb{R}^n$. Then the function $r \colon \mathbb{R}^n \to X$ is continuous, and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$, as required. ∎

8.2. The Kakutani Fixed Point Theorem

Theorem 8.6 (Kakutani's Fixed Point Theorem)

Let X be a non-empty, compact and convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\Phi \colon X \rightrightarrows X$ be a correspondence mapping X into itself. Suppose that the graph of the correspondence Φ is closed and that $\Phi(\mathbf{x})$ is non-empty and convex for all $\mathbf{x} \in X$. Then there exists a point $\mathbf{x}^*$ of X that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Proof

There exists a continuous map $r\colon \mathbb{R}^n \to X$ from $\mathbb{R}^n$ to X with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$. (see Proposition 8.5). Let Δ be an n-dimensional simplex chosen such that $X \subset \Delta$, and let $\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$ for all $\mathbf{x} \in \Delta$. If $\mathbf{x}^* \in \Delta$ satisfies $\mathbf{x}^* \in \Psi(\mathbf{x}^*)$ then $\mathbf{x}^* \in X$ and $r(\mathbf{x}^*) = \mathbf{x}^*$, and therefore $\mathbf{x} \in \Phi(\mathbf{x}^*)$. It follows that the result in the general case follows from that in the special case in which the closed bounded convex subset X of $\mathbb{R}^n$ is an n-dimensional simplex.

Thus let Δ be an n-dimensional simplex contained in $\mathbb{R}^n$, and let $\Phi\colon \Delta \rightrightarrows \Delta$ be a correspondence with closed graph, where $\Phi(\mathbf{x})$ is a non-empty closed convex subset of Δ for all $\mathbf{x} \in X$. We must prove that there exists some point $\mathbf{x}^*$ of Δ with the property that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let K be the simplicial complex consisting of the n-simplex Δ together with all its faces, and let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j. Then $|K^{(j)}| = \Delta$ for all positive integers j. Now $\Phi(\mathbf{v})$ is non-empty for all vertices $\mathbf{v}$ of $K^{(j)}$. Now any function mapping the vertices of a simplicial complex into a Euclidean space extends uniquely to a piecewise linear map defined over the polyhedron of that simplicial complex (Proposition 6.10). Therefore there exists a sequence $f_1, f_2, f_3, \ldots$ of continuous functions mapping the simplex Δ into itself such that, for each positive integer j, the continuous map $f_j \colon \Delta \to \Delta$ is piecewise linear on the simplices of $K^{(j)}$ and satisfies $f_j(\mathbf{v}) \in \Phi(\mathbf{v})$ for all vertices $\mathbf{v}$ of $K^{(j)}$.

Now it follows from the Brouwer Fixed Point Theorem Theorem 7.3 that, for each positive integer j, there exists $\mathbf{z}_j \in \Delta$ for which $f_j(\mathbf{z}_j) = \mathbf{z}_j$. For each positive integer j, there exist vertices

$$\mathbf{v}_{0,j}, \mathbf{v}_{1,j}, \ldots, \mathbf{v}_{n,j}$$

of $K^{(j)}$ spanning a simplex of K and non-negative real numbers $t_{0,j}, t_{1,j}, \ldots, t_{n,j}$ satisfying $\sum_{i=1}^{n} t_{i,j} = 1$ such that

$$\mathbf{z}_j = \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}$$

for all positive integers j. Let $\mathbf{y}_{i,j} = f_j(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j. Then $\mathbf{y}_{i,j} \in \Phi(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j.

The function f_j is piecewise linear on the simplices of $K^{(j)}$. It follows that

$$\begin{aligned} \sum_{i=0}^{m} t_{i,j}\mathbf{v}_{i,j} &= \mathbf{z}_j = f_j(\mathbf{z}_j) = f_j\left(\sum_{i=0}^{m} t_{i,j}\mathbf{v}_{i,j}\right) \\ &= \sum_{i=0}^{m} t_{i,j} f_j(\mathbf{v}_{i,j}) = \sum_{i=0}^{m} t_{i,j}\mathbf{y}_{i,j} \end{aligned}$$

for all positive integers j. Also $|\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| \leq \mu(K^{(j)})$ for $i = 0, 1, \ldots, n$ and for all positive integers j, where $\mu(K^{(j)})$ denotes the mesh of the simplicial complex $K^{(j)}$ (i.e., the length of the longest side of that simplicial complex). Moreover $\mu(K^j) \to 0$ as $j \to +\infty$ (see Lemma 6.8). It follows that

$$\lim_{j\to+\infty} |\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| = 0.$$

Now the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) ensures the existence of points $\mathbf{x}^*$, $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$ of the simplex Δ, non-negative real numbers $t_0, t_1, \ldots, t_n$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of positive integers, where

$$m_1 < m_2 < m_3 < \cdots ,$$

such that

$$\begin{aligned} \mathbf{x}^* &= \lim_{j \to +\infty} \mathbf{v}_{0,m_j}, \\ \mathbf{y_i} &= \lim_{j \to +\infty} \mathbf{y}_{i,m_j} \quad (0 \leq i \leq n), \\ t_i &= \lim_{j \to +\infty} t_{i,m_j} \quad (0 \leq i \leq n). \end{aligned}$$

Now

$$|\mathbf{v}_{i,m_j} - \mathbf{x}^*| \leq |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| + |\mathbf{v}_{0,m_j} - \mathbf{x}^*|$$

for $i = 0, 1, \ldots, n$ and for all positive integers j. Moreover $\lim_{j\to+\infty} |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| = 0$ and $\lim_{j\to+\infty} |\mathbf{v}_{i,m_j} - \mathbf{x}^*| = 0$. It follows that $\lim_{j\to+\infty} \mathbf{v}_{i,m_j} = \mathbf{x}^*$ for $i = 0, 1, \ldots, n$. Also

$$\sum_{i=0}^{n} t_i = \lim_{j\to+\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \right) = 1.$$

It follows that

$$\begin{aligned}
\lim_{j\to+\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) &= \sum_{i=0}^{n} \left(\lim_{j\to+\infty} t_{i,m_j} \right) \left(\lim_{j\to+\infty} \mathbf{v}_{i,m_j} \right) \\
&= \sum_{i=0}^{n} t_i \mathbf{x}^* = \mathbf{x}^*.
\end{aligned}$$

But we have also shown that $\sum_{i=0}^{m} t_{i,j}\mathbf{y}_{i,j} = \sum_{i=0}^{m} t_{i,j}\mathbf{v}_{i,j}$ for all positive integers j. It follows that

$$\sum_{i=0}^{m} t_i\mathbf{y}_i = \lim_{j\to+\infty}\left(\sum_{i=0}^{m} t_{i,m_j}\mathbf{y}_{i,m_j}\right) = \lim_{j\to+\infty}\left(\sum_{i=0}^{m} t_{i,m_j}\mathbf{v}_{i,m_j}\right) = \mathbf{x}^*.$$

Next we show that $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, n$. Now

$$(\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

for all positive integers j, and the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed. It follows that

$$(\mathbf{x}^*, \mathbf{y}_i) = \lim_{j \to +\infty} (\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

and thus $\mathbf{y}_i \in \Phi(\mathbf{x}_*)$ for $i = 0, 1, \ldots, m$ (see Proposition 4.6).

It follows from the convexity of $\Phi(\mathbf{x}^*)$ that

$$\sum_{i=0}^{q} t_i \mathbf{y}_* \in \Phi(\mathbf{x}^*).$$

(see Lemma 8.2). But $\sum_{i=0}^{q} t_i \mathbf{y}_* = \mathbf{x}^*$. It follows that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$, as required. ∎