MA3486 Fixed Point Theorems and Economic Equilibria
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Theorem 7.3

(Brouwer Fixed Point Theorem) Let X be a subset of a Euclidean space that is homeomorphic to the closed n-dimensional ball E^n, where

$$E^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f\colon X \to X$ mapping the set X into itself has at least one fixed point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

Proof

The closed n-dimensional ball E^n is itself homeomorphic to an n-dimensional simplex Δ. It follows that there exists a homeomorphism $h\colon X \to \Delta$ mapping the set X onto the simplex Δ. Then the continuous map $f\colon X \to X$ determines a continuous map $g\colon \Delta \to \Delta$, where $g(h(\mathbf{x}) = h(f(\mathbf{x}))$ for all $\mathbf{x} \in X$. Suppose that it were the case that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in X$. Then $g(\mathbf{z}) \neq \mathbf{z}$ for all $\mathbf{z} \in \Delta$. There would then exist a well-defined continuous map $r\colon \Delta \to \partial\Delta$ mapping each point $\mathbf{z}$ of Δ to the unique point $r(\mathbf{z})$ of the boundary $\partial\Delta$ of Δ at which the half line starting at $g(\mathbf{z})$ and passing through $\mathbf{z}$ intersects $\partial\Delta$. Then $r\colon \Delta \to \partial\Delta$ would be continuous, and $r(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \partial\Delta$. However Proposition 7.2 guarantees that there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with these properties. Therefore the map f must have at least one fixed point, as required. ∎

Corollary 7.4

Let

$$\Delta = \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} p_i = 1\},$$

let $\mathbf{z}\colon \Delta \to \mathbb{R}^n$ *be a continuous function mapping* Δ *into* $\mathbb{R}^n$*, and let*

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$*. Suppose that* $\mathbf{p}.\mathbf{z}(\mathbf{p}) \leq 0$ *for all* $\mathbf{p} \in \Delta$*. Then there exists* $\mathbf{p}^* \in \Delta$ *such that* $z_i(\mathbf{p}^*) \leq 0$ *for* $i = 1, 2, \ldots, n$*.*

Proof

Let $\mathbf{v}\colon \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi\colon \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem (Theorem 7.3) ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda \mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^*.\mathbf{z}(\mathbf{p}^*) > 0$, contradicting the requirement that $\mathbf{p}.\mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required. ∎