MA3486 Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 21 (March 11, 2016)

David R. Wilkins

7. The Brouwer Fixed Point Theorem

7.1. Sperner's Lemma

Definition

Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

- for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
- if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

Lemma 7.1

(Sperner's Lemma) Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.

Proof

Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2\big(N(0, 1, \ldots, n-1) - M\big),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required. $\blacksquare$

7.2. Proof of Brouwer's Fixed Point Theorem

Proposition 7.2

Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

Proof

Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem (Theorem 6.16) that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

7. The Brouwer Fixed Point Theorem (continued)

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma (Lemma 7.1) guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. $\blacksquare$