MA3486 Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 20 (March 10, 2016)

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Definition

Let K be a simplicial complex in n-dimensional Euclidean space. A function $f\colon |K| \to \mathbb{R}^m$ mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ is said to be *piecewise linear* on each simplex of K if

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i)$$

for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K, and for all non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum\limits_{i=0}^{q} t_i = 1$.

Lemma 6.9

Let K be a simplicial complex in n-dimensional Euclidean space, and let $f: |K| \to \mathbb{R}^m$ be a function mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ that is piecewise linear on each simplex of K. Then $f: |K| \to \mathbb{R}^m$ is continuous.

Proof

The definition of piecewise linear functions ensures that the restriction of $f: |K| \to \mathbb{R}^m$ to each simplex of K is continuous on that simplex. The result therefore follows from Lemma 6.1. ∎

Proposition 6.10

Let K be a simplicial complex in n-dimensional Euclidean space and let $\alpha\colon \mathrm{Vert}(K) \to \mathbb{R}^m$ be a function mapping the set $\mathrm{Vert}(K)$ of vertices of K into m-dimensional Euclidean space $\mathbb{R}^m$. Then there exists a unique function $f\colon |K| \to \mathbb{R}^m$ defined on the polyhedron $|K|$ of K that is piecewise linear on each simplex of K and satisfies $f(\mathbf{v}) = \alpha(\mathbf{v})$ for all vertices $\mathbf{v}$ of K.

Proof

Given any point $\mathbf{x}$ of K, there exists a unique simplex of K whose interior contains the point $\mathbf{x}$ (Proposition 6.4). Let the vertices of this simplex be $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$, where $p \leq n$. Then there exist uniquely-determined strictly positive real numbers $t_0, t_1, \ldots, t_p$ satisfying $\sum_{i=0}^{p} t_i = 1$ for which $\mathbf{x} = \sum\limits_{i=0}^{p} t_i \mathbf{v}_i$. We then define $f(\mathbf{x})$ so that

$$f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i).$$

Defining $f(\mathbf{x})$ in this fashion at each point $\mathbf{x}$ of $|K|$, we obtain a function $f\colon |K| \to \mathbb{R}^m$ mapping Δ into $\mathbb{R}^m$.

Now let $\mathbf{x} \in \sigma$ for some q-simplex of K. We can order the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ so that the point $\mathbf{x}$ belongs to the interior of the face of σ spanned by $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$ where $p \leq q$. Let $t_1, t_2, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the simplex σ. Then $\mathbf{x} = \sum\limits_{i=0}^{q} t_i \mathbf{v}_i$, where $t_i > 0$ for those integers i satisfying $0 \leq i \leq p$, $t_i = 0$ for those integers i (if any) satisfying $p < i \leq q$, and $\sum\limits_{i=0}^{p} t_i = \sum\limits_{i=0}^{q} t_i = 1$. Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i).$$

The result follows. ∎

Corollary 6.11

Let K be a simplicial complex in $\mathbb{R}^n$ and let L be simplicial complexes in $\mathbb{R}^m$, where m and n are positive integers, and let $\varphi\colon \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ be a function mapping vertices of K to vertices of L. Suppose that

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K. Then there exists a unique continuous map $\overline{\varphi}\colon |K| \to |L|$ mapping the polyhedron $|K|$ of K into the polyhedron $|L|$ of L that is piecewise linear on each simplex of K and satisfies $\overline{\varphi}(\mathbf{v}) = \varphi(\mathbf{v})$ for all vertices $\mathbf{v}$ of K. Moreover this function maps the interior of a simplex of K spanned by vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ into the interior of the simplex of L spanned by $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

Proof

It follows from Proposition 6.10 that there is a unique piecewise linear function $f\colon |K| \to \mathbb{R}^m$ that satisfies $f(\mathbf{v}) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in \mathrm{Vert}(K)$. We show that $f(|K|) \subset |L|$.

Let

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

be vertices of a simplex σ of K. Then

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L. Let τ be the simplex of L spanned by these vertices of L, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ be the vertices of τ. Then, for each integer j between 1 and r, let u_j be the sum of those t_i for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$.

Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i \varphi(\mathbf{v}_i) = \sum_{j=0}^{r} u_j \mathbf{w}_j$$

and $\sum_{j=0}^{r} u_j = 1$. It follows that $f(\sigma) \subset \tau$. Moreover, given any integer j between 1 and r, there exists at least one integer i between 1 and q for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$. It follows that $t_0, t_1, t_2, \ldots, t_q$ are all strictly positive, then $u_0, u_1, \ldots, u_r$ are also all strictly positive. Therefore the piecewise linear function f maps the interior of σ into the interior of τ.

We have already shown that $f: |K| \to \mathbb{R}^m$ maps each simplex of K into a simplex of L. Therefore there exists a uniquely-determined linear function $\overline{\varphi}: |K| \to |L|$ satisfying $\overline{\varphi}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in |K|$. The result follows. $\blacksquare$

6.7. Simplicial Maps

Definition

A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes K and L is a function $\varphi\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

It follows from Corollary 6.11 that simplicial map $\varphi\colon K \to L$ also induces in a natural fashion a continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Moreover it also follows from Corollary 6.11 that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

6.8. Simplicial Approximations

Definition

Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

Definition

Let X and Y be subsets of Euclidean spaces. Continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ from X to Y are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Lemma 6.12

Let K and L be simplicial complexes, let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of K and L, and let $s\colon K \to L$ be a simplicial approximation to the map f. Then there is a well-defined homotopy $H\colon |K| \times [0,1] \to |L|$, between the maps f and s, where

$$H(\mathbf{x}, t) = (1-t)f(\mathbf{x}) + ts(\mathbf{x})$$

for all $\mathbf{x} \in |K|$ and $t \in [0,1]$.

Proof

Let $\mathbf{x} \in |K|$. Then there is a unique simplex σ of L such that the point $f(\mathbf{x})$ belongs to the interior of σ. Then $s(\mathbf{x}) \in \sigma$. But, given any two points of a simplex embedded in some Euclidean space, the line segment joining those two points is contained within the simplex. It follows that $(1-t)f(\mathbf{x}) + ts(\mathbf{x}) \in |L|$ for all $\mathbf{x} \in K$ and $t \in [0,1]$. Thus the homotopy $H\colon |K| \times [0,1] \to |L|$ is a well-defined map from $|K| \times [0,1]$ to $|L|$. Moreover it follows directly from the definition of this map that $H(\mathbf{x},0) = f(\mathbf{x})$ and $H(\mathbf{x},1) = s(\mathbf{x})$ for all $\mathbf{x} \in |K|$ and $t \in [0,1]$. The map H is thus a homotopy between the maps f and s, as required. ∎

Definition

Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star neighbourhood* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

Lemma 6.13

Let K be a simplicial complex and let $\mathbf{x} \in |K|$. Then the star neighbourhood $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ is open in $|K|$, and $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$.

Proof

Every point of $|K|$ belongs to the interior of a unique simplex of K (Proposition 6.4). It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K. ∎

Proposition 6.14

A function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ *between the vertex sets of simplicial complexes* K *and* L *is a simplicial map, and a simplicial approximation to some continuous map* $f\colon |K| \to |L|$*, if and only if* $f\,(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L\,(s(\mathbf{v}))$ *for all vertices* $\mathbf{v}$ *of* K*.*

Proof

Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, because $\mathbf{x}$ is in the interior of σ (see Corollary 6.11). It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \mathrm{Vert}\,K \to \mathrm{Vert}\,L$ be a function with the property that $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}_j))$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \mathrm{Vert}\,K \to \mathrm{Vert}\,L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required. ∎

Corollary 6.15

If $s\colon K \to L$ and $t\colon L \to M$ are simplicial approximations to continuous maps $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$, where K, L and M are simplicial complexes, then $t \circ s\colon K \to M$ is a simplicial approximation to $g \circ f\colon |K| \to |M|$.

6.9. The Simplicial Approximation Theorem

Theorem 6.16

(Simplicial Approximation Theorem) Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then, for some sufficiently large integer j, there exists a simplicial approximation $s\colon K^{(j)} \to L$ to f defined on the jth barycentric subdivision $K^{(j)}$ of K.

Proof

The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ (Lemma 6.13) and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$.

Now the set $|K|$ is a closed bounded subset of a Euclidean space. It follows that there exists a Lebesgue number δ_L for the open cover consisting of the preimages of the stars of all the vertices of L (see Proposition 3.1). This Lebesgue number δ_L is a positive real number with the following property: every subset of $|K|$ whose diameter is less than δ_L is contained in the preimage of the star of some vertex $\mathbf{w}$ of L. It follows that every subset of $|K|$ whose diameter is less than δ_L is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$ (see Lemma 6.8). Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta_L$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta_L$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ_L. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. It follows directly from Proposition 6.14 that this is the desired simplicial approximation to f. ∎