MA3486 Fixed Point Theorems and Economic Equilibria
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We define (by induction on j) the jth barycentric subdivision $K^{(j)}$ of K to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.

Lemma 6.6

Let σ be a q-simplex and let τ be a face of σ. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of σ and τ respectively. If all the 1-simplices (edges) of σ have length not exceeding d for some $d > 0$ then

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

Proof

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ. Let $\mathbf{x}$ and $\mathbf{y}$ be points of σ. We can write $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. Now

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= \left| \sum_{i=0}^{q} t_i (\mathbf{x} - \mathbf{v}_i) \right| \leq \sum_{i=0}^{q} t_i |\mathbf{x} - \mathbf{v}_i| \\ &\leq \operatorname{maximum} \left(|\mathbf{x} - \mathbf{v}_0|, |\mathbf{x} - \mathbf{v}_1|, \ldots, |\mathbf{x} - \mathbf{v}_q| \right). \end{aligned}$$

Applying this result with $\mathbf{x} = \hat{\sigma}$ and $\mathbf{y} = \hat{\tau}$, we find that

$$|\hat{\sigma} - \hat{\tau}| \leq \mathrm{maximum}\left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|\right).$$

But

$$\hat{\sigma} = \frac{1}{q+1}\mathbf{v}_i + \frac{q}{q+1}\mathbf{z}_i$$

for $i = 0, 1, \ldots, q$, where $\mathbf{z}_i$ is the barycentre of the $(q-1)$-face of σ opposite to $\mathbf{v}_i$, given by

$$\mathbf{z}_i = \frac{1}{q}\sum_{j \neq i} \mathbf{v}_j.$$

Moreover $\mathbf{z}_i \in \sigma$. It follows that

$$|\hat{\sigma} - \mathbf{v}_i| = \frac{q}{q+1} |\mathbf{z}_i - \mathbf{v}_i| \leq \frac{qd}{q+1}$$

for $i = 1, 2, \ldots, q$, and thus

$$|\hat{\sigma} - \hat{\tau}| \leq \mathrm{maximum}\left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|\right) \leq \frac{qd}{q+1},$$

as required. ∎

Definition

The *mesh* $\mu(K)$ of a simplicial complex K is the length of the longest edge of K.

Lemma 6.7

Let K be a simplicial complex, and let n be the dimension of K. Let K' be the first barycentric subdivision of K. Then

$$\mu(K') \leq \frac{n}{n+1}\mu(K).$$

Proof

A 1-simplex of K' is of the form $(\hat{\tau}, \hat{\sigma})$, where σ is a q-simplex of K for some $q \leq n$ and τ is a proper face of σ. Then

$$|\hat{\tau} - \hat{\sigma}| \leq \frac{q}{q+1}\mu(K) \leq \frac{n}{n+1}\mu(K)$$

by Lemma 6.6, as required. ∎

Lemma 6.8

Let K be a simplicial complex, let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j, and let $\mu(K^{(j)})$ be the mesh of $K^{(j)}$. Then $\lim_{j\to+\infty} \mu(K^{(j)}) = 0$.

Proof

The dimension of all barycentric subdivisions of a simplicial complex is equal to the dimension of the simplicial complex itself. It therefore follows from Lemma 6.7 that

$$\mu(K^{(j)}) \leq \left(\frac{n}{n+1}\right)^j \mu(K).$$

The result follows. ∎