MA3486 Fixed Point Theorems and Economic Equilibria
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We shall denote by $\mathrm{Vert}\, K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition

Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

6.3. Barycentric Coordinates on a Simplex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j) \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent.

Definition

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$) are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates $t_0, t_1, \ldots, t_q$ of a point of a q-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

Example

Consider the triangle τ in $\mathbb{R}^3$ with vertices at **i**, **j** and **k**, where

$$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0) \quad \text{and} \quad \mathbf{k} = (0,0,1).$$

Then

$$\tau = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x, y, z \le 1 \text{ and } x + y + z = 1\}.$$

The barycentric coordinates on this triangle τ then coincide with the Cartesian coordinates x, y and z, because

$$(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$$

for all $(x,y,z) \in \tau$.

Example

Consider the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. This triangle is the set

$$\{(x,y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } x + y \leq 1.\}.$$

The barycentric coordinates of a point (x,y) of this triangle are t_0, t_1 and t_2, where

$$t_0 = 1 - x - y, \quad t_1 = x \quad \text{and} \quad t_2 = y.$$

Example

Consider the triangle in $\mathbb{R}^2$ with vertices at $(1,2)$, $(3,3)$ and $(4,5)$. Let t_0, t_1 and t_2 be the barycentric coordinates of a point (x,y) of this triangle. Then t_0, t_1, t_2 are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x,y) = (1 - t_1 - t_2)(1,2) + t_1(3,3) + t_2(4,5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.

6.4. The Interior of a Simplex

Definition

The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Lemma 6.2

Let σ be a q-simplex in some Euclidean space with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Let $\mathbf{x}$ be a point of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, so that $t_j \geq 0$ for $j = 0, 1, \ldots, q$, $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, and $\sum_{j=0}^{q} t_j = 1$. Then the point $\mathbf{x}$ belongs to the interior of σ if and only if $t_j > 0$ for $j = 0, 1, \ldots, q$.

Proof

The point $\mathbf{x}$ belongs to the face of σ spanned by vertices $\mathbf{v}_{j_0}, \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r}$, where $0 \leq j_0 < j_1 < \cdots < j_r \leq q$, if and only if $t_j = 0$ for all integers j between 0 and q that do not belong to the set $\{j_0, j_1, \ldots, j_r\}$. Thus the point $\mathbf{x}$ belongs to a proper face of the simplex σ if and only if at least one of the barycentric coordinates t_j of that point is equal to zero. The result follows. ∎

Example
A 0-simplex consists of a single vertex $\mathbf{v}$. The interior of that 0-simplex is the vertex $\mathbf{v}$ itself.

Example
A 1-simplex is a line segment. The interior of a line segment in a Euclidean space $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$ is

$$\{(1-t)\,\mathbf{v} + t\,\mathbf{w} : 0 < t < 1\}.$$

Thus the interior of the line segment consists of all points of the line segement that are not endpoints of the line segment.

Example

A 2-simplex is a triangle. The interior of a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 < r, s, t < 1 \text{ and } r + s + t = 1\}.$$

The interior of this triangle consists of all points of the triangle that do not lie on any edge of the triangle.

Remark

Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. If $k > q$ then the interior of the simplex (defined according to the definition given above) will not coincide with the topological interior determined by the usual topology on $\mathbb{R}^k$. Consider for example a triangle embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The interior of the triangle (defined according to the definition given above) consists of all points of the triangle that do not lie on any edge of the triangle. But of course no three-dimensional ball of positive radius centred on any point of that triangle is wholly contained within the triangle. It follows that the topological interior of the triangle is the empty set when that triangle is considered as a subset of three-dimensional space $\mathbb{R}^3$.