MA3486 Fixed Point Theorems and Economic Equilibria
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We describe another proof of the Berge Maximum Theorem using the characterization of compact-valued upper hemicontinuous correspondences using sequences established in Proposition 4.17 and the characterization of lower hemicontinuous correspondences using sequences established in Proposition 4.20. First we introduce some terminology.

Definition

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Let $(\mathbf{x}_j : j \in \mathbb{N})$ be a sequence of points of the domain X of the correspondence. We say that an infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in the codomain of the correspondence is a *companion sequence* for $(\mathbf{x}_j)$ with respect to the correspondence Φ if $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the continuity properties of $\Phi\colon X \rightrightarrows Y$ can be characterized in terms of companion sequences with respect to Φ as follows:—

- the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y, that companion sequence has a subsequence that converges to a point of $\Phi(\mathbf{p})$ (Proposition 4.17);
- the correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists a companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y converging to the point $\mathbf{q}$. (Proposition 4.20).

Proof of Theorem 4.24 using Companion Sequences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper and lower hemicontinuous and that also has the property that $\Phi(\mathbf{x})$ is non-empty and compact for all $\mathbf{x} \in X$. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M\colon X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. We must prove that $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

It follows from the continuity of $f: X \times Y \to \mathbb{R}$ that $M(\mathbf{x})$ is closed in $\Phi(\mathbf{x})$ for all $\mathbf{x} \in X$. It also follows from the Extreme Value Theorem (Theorem 2.20) that $M(\mathbf{x})$ is non-empty for all $\mathbf{x}$.

Let $(\mathbf{x}_j, j \in \mathbb{N})$ be a sequence in X which converges to a point $\mathbf{p}$ of X, and let $(\mathbf{y}_j^* : j \in \mathbb{N})$ be a companion sequence of $(\mathbf{x}_j)$ with respect to the correspondence M. Then, for each positive integer j, $\mathbf{y}_j^* \in \Phi(\mathbf{x}_j)$ and

$$f(\mathbf{x}_j, \mathbf{y}_j^*) \geq f(\mathbf{x}_j, \mathbf{y})$$

for all $\mathbf{y} \in \Phi(\mathbf{x}_j)$. Now the correspondence Φ is compact-valued and upper hemicontinuous. It follows from Proposition 4.17 that there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{N})$ that converges to an element $\mathbf{q}$ of $\Phi(\mathbf{q})$. Let that subsequence be the sequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ whose members are

$$\mathbf{y}_{k_1}^*, \mathbf{y}_{k_2}^*, \mathbf{y}_{k_3}^*, \ldots,$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}_{k_j}^*$.

We show that $\mathbf{q} \in M(\mathbf{p})$. Let $\mathbf{r} \in \Phi(\mathbf{p})$. The correspondence $\Phi\colon X \to Y$ is lower hemicontinuous. It follows that there exists a companion sequence $(\mathbf{z}_j : j \in N)$ to $(\mathbf{x}_j : j \in N)$ with respect to the correspondence Φ that converges to $\mathbf{r}$ (Proposition 4.20). Then

$$\lim_{j\to+\infty} \mathbf{y}^*_{k_j} = \mathbf{q} \quad \text{and} \quad \lim_{j\to+\infty} \mathbf{z}_{k_j} = \mathbf{r}.$$

It follows from the continuity of $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) = f(\mathbf{p}, \mathbf{q}) \quad \text{and} \quad \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Now

$$f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j})$$

for all positive integers j, because $\mathbf{y}^*_{k_j} \in M(\mathbf{x}_{k_j})$. It follows that

$$f(\mathbf{p}, \mathbf{q}) = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Thus $f(\mathbf{p}, \mathbf{q}) \geq f(\mathbf{p}, \mathbf{r})$ for all $\mathbf{r} \in \Phi(\mathbf{p})$. It follows that $\mathbf{q} \in M(\mathbf{p})$.

We have now shown that, given any sequence $(\mathbf{x}_j : j \in \mathbb{R})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j^* : j \in \mathbb{R})$ with respect to the correspondence M, there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{R})$ that converges to a point of $M(\mathbf{x})$. It follows that the correspondence $M\colon X \to Y$ is compact-valued and upper hemicontinuous at the point $\mathbf{p}$ (Proposition 4.17).

It remains to show that the function $m\colon X \to \mathbb{R}$ is continuous at the point $\mathbf{p}$, where $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ for all $\mathbf{x} \in X$ and $\mathbf{y}^* \in M(\mathbf{x})$. Let $(\mathbf{x}_j : j \in \mathbb{R})$ be an infinite sequence converging to the point $\mathbf{p}$, and let $v_j = m(\mathbf{x}_j)$ for all positive integers j. Then there exists an infinite sequence Let $(\mathbf{y}_j^* : j \in \mathbb{R})$ in Y that is a companion sequence to $(\mathbf{x}_j)$ with respect to the correspondence M. Then $\mathbf{y}_j^* \in M(\mathbf{x}_j)$ and therefore $v_j = f(\mathbf{x}_j, \mathbf{y}_j^*)$ for all positive integers j. Now the correspondence $M\colon X \rightrightarrows Y$ has been shown to be compact-valued and upper hemicontinuous. There therefore exists a subsequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ of $(\mathbf{y}_j)$ that converges to a point $\mathbf{q}$ of $M(\mathbf{p})$. It then follows from the continuity of the function $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} m(\mathbf{x}_{k_j}) = \lim_{j\to+\infty} v_{k_j} = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) = f(\mathbf{p}, \mathbf{q}) = m(\mathbf{p}).$$

Now the result just proved can be applied with any subsequence of $(\mathbf{x}_j : j \in \mathbb{N})$ in place of the original sequence. It follows that *every subsequence* of of $(v_j : j \in \mathbb{R})$ itself has a subsequence that converges to $m(\mathbf{p})$.

Let some positive real number ε be given. Suppose that there did not exist any positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. Then there would exist infinitely many positive integers j for which $|v_j - m(\mathbf{p})| \geq \varepsilon$. It follows that there would exist some subsequence

$$v_{l_1}, v_{l_2}, v_{l_3}, \ldots$$

of $v_1, v_2, v_3, \ldots$ with the property that $|v_{l_j} - m(\mathbf{p})| \geq \varepsilon$ for all positive integers j. This subsequence would not in turn contain any subsequences converging to the point $m(\mathbf{p})$.

But we have shown that every subsequence of $(v_j : j \in \mathbb{N})$ contains a subsequence converging to $m(\mathbf{p})$. It follows that there must exist some positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. We conclude from this that $\lim_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$.

We have shown that if $(\mathbf{x}_j : j \in \mathbb{N})$ is an infinite sequence in X and if $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ then $\lim_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$. It follows that the function $m\colon X \to \mathbb{R}$ is continuous at $\mathbf{p}$. This completes the proof of Berge's Maximum Theorem. ∎

Remark

In 1959, the French mathematician Claude Berge published a book entitled *Espaces topologiques: fonctions multivoques* (Dunod, Paris, 1959). This book was subsequently translated into English by E.M. Patterson, and the translation was published with the title *Topological spaces, including a treatment of multi-valued functions, vector spaces and convexity* (Oliver and Boyd, Edinburgh and London, 1963).

Claude Berge had completed his Ph.D. at the University of Paris in 1953, supervised by the differential geometer and mathematical physicist *André Lichnerowicz*. His thesis was entitled *Sur une théorie ensembliste des jeux alternatifs*, and a paper of that name was published by him (*J. Math. Pures Appl.* **32** (1953), 129–184). He subsequently published *Théorie Générale des Jeux à N Personnes* (Gauthier Villars, Paris, 1957). The title translates as "General theory of *n*-person games".

Claude Berge was Professor at the Institute of Statistics at the University of Paris from 1957 to 1964, and subsequently directed the International Computing Center in Rome. Following his early work in game theory, his research developed in the fields of combinatorics and graph theory.

The preface of the 1959 book, *Espaces topologiques: fonctions multivoques*, includes a passage translated by E.M. Patterson as follows:—

> *In Set Topology, with which we are concerned in this book, we study sets in topological spaces and topological vector spaces; whenever these sets are colletions of n-tuples or classes of functions, we recover well-known results of classical analysis.*

But the role of topology does not stop there; the majority of text-books seem to ignore certain problems posed by the calculus of probabilities, the decision functions of statistics, linear programming, cybernetics, economics; thus, in order to provide a topological tool which is of equal interest to the student of pure mathematics and the student of applied mathematics, we have felt it desirable to include a systematic devcelopment of the properties of multi-valued functions.

The following theorem is included in *Espaces topologiques* by Claude Berge (Chapter 6, Section 3, page 122):—

> **Théorème du maximum.** — *Si $\varphi(y)$ est une fonction numérique continue dans Y, et si Γ est un application continue de X dans Y telle que $\Gamma x \neq \emptyset$ pour tout x,*
>
> $$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$
>
> *est une fonction numérique continue dans X, et*
>
> $$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$
>
> *est une application u.s.c. de X dans Y.*

This theorem is translated by E.M. Patterson as follows (*Topological Spaces*, Claude Berge, translated by E.M. Patterson, Oliver and Boyd, Edinburgh, 1963, in Chapter 6, Section 3, page 116):—

> **Maximum Theorem** — *If* φ *is a continuous numerical function in* Y *and* Γ *is a continuous mapping of* X *into* Y *such that, for each* x, $\Gamma x \neq \emptyset$, *then the numerical function* M *defined by*
>
> $$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$
>
> *is continuous in* X *and the mapping* Φ *defined by*
>
> $$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$
>
> *is an u.s.c. mapping of* X *into* Y.

In this context X and Y are Hausdorff topological spaces. Indeed in Chapter 4, Section 5 of *Espaces topologiques*, Berge introduces the concept of a *separated* (or *Hausdorff*) space and then, after some discussion of separation properties, makes that statement translated by E.M. Patterson as follows:—

> *In what follows all the topological spaces which we consider will be assumed to be separated.*

It seems that, in the original statement, the objective function φ was required to be a continuous function on Y, but the first sentence of the proof of the "Maximum Theorem" notes that φ is continuous on $X \times Y$. A "mapping" in Berge is a correspondence. A mapping (or correspondence) is said by Berge to be "upper semi-continuous" when it is both compact-valued and upper hemicontinuous; a mapping is said by Berge to be "lower semi-continuous" when it is lower hemicontinuous.

Berge's proof of the *Théorème du maximum* is just one short paragraph, but requires the work of earlier theorems. We discuss his proof using the terminology adopted in these lectures. In Theorem 1 of Chapter 6, Section 4, Berge shows that if the correpondence $\Gamma\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \leq M(x_0) + \varepsilon$ throughout some open neighbourhood of the point x_0. (This result can be compared with Lemma 4.22 and the first proof of Theorem 4.24 presented in these notes.) In Theorem 2 of Chapter 6, Section 4, Berge shows that if the correspondence Γ is lower hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \geq M(x_0) - \varepsilon$ throughout some open neighbourhood of the point x_0.

(This result can be compared with Lemma 4.23 and the first proof of Theorem 4.24 presented in these notes.) These two results ensure that if Γ is compact-valued, everywhere non-empty and both upper and lower hemicontinuous then the function function M is continuous on X. In Theorem 7 of Chapter 6, Section 1, Berge had proved that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact valued and upper hemicontinuous (see Proposition 4.21 of these notes). Berge completes the proof of the *Théorème du maximum* by putting these results together in a fashion to obtain a proof (in the contexts of correspondences between Hausdorff topological spaces) similar in structure to the first proof of Theorem 4.24 presented in these notes.

The definitions of "upper-semicontinuous" and "lower-semicontinuous" mappings (i.e., correspondences) Given by Claude Berge at the beginning of Chapter VI are accompanied by a footnote translated by E.M. Patterson as follows (C. Berge, translated E.M. Patterson, *Topological Spaces*, *loc. cit.*, p. 109):—

> *The two kinds of semi-continuity of a multivalued function were introduced independently by Kuratowski (Fund. Math.* **18***, 1932, p.148) and Bouligand (Ens. Math., 1932, p. 14). In general, the definitions given by different authors do not coincide whenever we deal with non-compact spaces (at least for upper semi-continuity, which is the more important from the point of view of applications). The definitions adopted here, which we have developed elsewhere (C. Berge, Mém. Sc. Math. 138), enable us to include the case when the image of a point* x *can be empty.*

In 1959, the year in which Claude Berge published *Espaces topologiques*, Gérard Debreu published his influential monograph *Theory of value: an axiomatic analysis of economic equilibrium* (Cowles Foundation Monographs 17, 1959). Section 1.8 of Debreu's monograph discusses "continuous correspondences", developing the theory of correspondences φ from S to T, where S is a subset of $\mathbb{R}^m$ and T is a compact subset of $\mathbb{R}^n$. Debreu also requires correspondences to be non-empty-valued. In consequence of these conventions, closed-valued correspondences from S to T must necessarily be compact-valued. Also a correspondence from S to T is upper hemicontinuous if and only if its graph is closed (see Propositions 4.11 4.12 of these notes).

In the notes to Chapter 1 of the *Theory of Value*, Debreu notes that "a study of the *continuity of correspondences* from a topological space to a topological space will be found in C. Berge [1], Chapter 6". The reference is to *Espace Topologiques*.

According to Debreu, the correspondence φ is *upper semicontinuous* at the point x^0 if the following condition is satisfied:

> "$x^q \to x^0$, $y^q \in \varphi(x^q)$, $y^q \to y^0$" *implies* "$y^0 \in \varphi(x^0)$".

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence has closed graph. Thus Debreu's definition is in accordance with the definition of *upper hemicontinuity* for those correspondences, and only those correspondences, where the codomain of the correspondence is a compact subset of a Euclidean space. Indeed Debreu notes the following in Section 1.8 of the *Theory of Value*:—

> *"(1) The correspondence φ is upper semicontinuous on S if and only if its graph is closed in $S \times T$."*

Again according to Debreu, the correspondence φ is *lower semicontinuous* at the point x^0 if the following condition is satisfied:

> "$x^q \to x^0$, $y^0 \in \varphi(x^0)$" *implies* "*there is* (y^q) *such that* $y^q \in \varphi(x^q)$, $y^q \to y^0$".

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence is lower hemicontinuous (in accordance with the definitions adopted in those notes, see Proposition 4.20 of these notes).

A correspondence from S to T is said by Debreu to be continuous if it is both upper semicontinuous and lower semicontinuous according to his definitions.

Debreu discusses Berge's Maximum Theorem, in the context of a correspondence φ from a subset S of $\mathbb{R}^m$ to a compact subset T of $\mathbb{R}^n$, as follows (*Theory of Value*, Section 1.8, page 19):—

> *The interest of these concepts for economics lies, in particular, in the interpretations of an element x of S as the environment of a certain agent, of T as the set of actions a priori available to him, and of $\varphi(x)$ (assumed here to be closed for every x in S) as the subset of T to which his choice is actually restricted by the environment x. Let f be a continuous real-valued function on $S \times T$, and interpret $f(x, y)$ as the gain for that agent when his environment is x and his action y. Given x, one is interested in the elements of $\varphi(x)$ which maximize f (now a function of y alone) on $\varphi(x)$; they form a set $\mu(x)$. What can be said about the continuity of the correspondence μ from S to T?*

One is also interested in $g(x)$, the value of the maximum of f on $\phi(x)$ for a given x. What can be said about the continuity of the real-valued function g on S? An answer to these two questions is given by the following result (the proof of the continuity of g should not be attempted).

(4) If f is continuous on $S \times T$, and if φ is continuous at $x \in S$, then μ is upper semicontinuous at x, and g is continuous on x.

(Note that, because the set T is compact and Debreu requires correspondences to be non-empty valued, the conventions adopted by Debreu in his *Theory of Value* ensure that if φ is an "upper semicontinuous" correspondence from a set S to a compact set T, where S and T are subsets of Euclidean spaces, then $\varphi(x)$ will necessarily be a non-empty compact subset of T.)

The book *Infinite dimensional analysis: a hitchhiker's guide* by Charalambos D. Aliprantis and Kim C. Border (2nd edition, Springer-Verlag, 1999) discusses the theory of continuous correspondences between topological spaces (Chapter 16). Berge's Maximum Theorem is stated and proved, in the context of correspondences between topological spaces, as Theorem 16.31 (p. 539). The definitions of upper hemicontinuity and lower hemicontinuity for correspondences are consistent with the definitions adopted in these lecture notes. These definitions are accompanied by the following footnote:—

> *J. C. Moore [. . .] identifies five slightly different definitions of upper semicontinuity in use by economists, and points out some of the differences for compositions, etc. T. Ichiishi [. . .] and E. Klein and A. C. Thompson [. . .] also give other notions of continuity.*

The book *Mathematical Methods and Models for Economists* by Angel de la Fuente (Cambridge University Press, 2000) includes a section on continuity of correspondences between subsets of Euclidean spaces (Chapter 2, Section 11). The definitions of upper and lower hemicontinuity adopted there are consistent with those given in these lecture notes. The sequential characterization of compact-valued upper hemicontinuous correspondences in terms of companion sequences (Proposition 4.17 of these lecture notes) is stated and proved as Theorem 11.2 of Chapter 2 of Angel de la Fuente's textbook. Similarly the sequencial characterization of lower hemicontinuous correspondences in terms of companion sequences Proposition 4.20 is stated and proved as Theorem 11.3 of that textbook.

Theorem 11.6 in Chapter 2 of that textbook covers the result that a closed-valued upper hemicontinuous correspondence has a closed graph (see Proposition 4.11) and the result that a correspondence with closed graph whose codomain is compact is upper hemicontinuous (see Proposition 4.12). The result that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact-valued and upper hemicontinuous (see Proposition 4.21) is Theorem 11.7 in Chapter 2 of the textbook by Angel de la Fuente. Berge's Maximal Theorem is Theorem 2.1 in Chapter 7 of that textbook. The proof is based on the use of the sequential characterizations of upper and lower hemicontinuity in terms of existence and properties of companion sequences.