MA3486 Fixed Point Theorems and Economic Equilibria
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Corollary 4.19

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let K be a compact subset of X. Then

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{x} \in K \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is a compact subset of $X \times Y$.

Proof

Let $\mathcal{V}$ be an open cover of L where

$$L = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{x} \in K \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}$$

For each $\mathbf{p} \in K$ let

$$L_{\mathbf{p}} = \{(\mathbf{p}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{p})\} = \{(\mathbf{p}, \mathbf{y}) : (\mathbf{p}, \mathbf{y}) \in L\}.$$

Then $L_{\mathbf{p}}$ is a compact subset of $X \times Y$ for all $\mathbf{p} \in K$. (Indeed this set is the image of the compact set $\Phi(\mathbf{p})$ under the continuous function that sends each point $\mathbf{y}$ of $\Phi(\mathbf{p})$ to $(\mathbf{p}, \mathbf{y})$, and any continuous function maps compact sets to compact sets.) It follows that, for each point $\mathbf{p}$ of K, there is some finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers $L_{\mathbf{p}}$.

Let $U_{\mathbf{p}}$ be the union of the sets belonging to the collection $\mathcal{W}_{\mathbf{p}}$. Then $U_{\mathbf{p}}$ is an open subset of $X \times Y$. Let

$$N_{\mathbf{p}} = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U_{\mathbf{p}} \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{p} \in K$. It then follows from Proposition 4.18 that that $N_{\mathbf{p}}$ is open in X for all $\mathbf{p} \in K$. Moreover the definition of $N_{\mathbf{p}}$ ensures that

$$\{(\mathbf{x}, \mathbf{y}) \in L : \mathbf{x} \in N_{\mathbf{p}}\}$$

is covered by the finite subcollection $\mathcal{W}_{\mathbf{p}}$ of the given open cover $\mathcal{V}$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers L. The result follows. $\blacksquare$

4.4. A Criterion characterizing Lower Hemicontinuity

Proposition 4.20

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

in X for which $\lim\limits_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of F such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$.

Proof

First suppose that $\Phi\colon X \to Y$ is lower hemicontinuous at some point $\mathbf{p}$ of X. Let $\mathbf{q} \in \Phi(\mathbf{p})$, and let some positive number ε be given. Then the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centred on the point $\mathbf{q}$ is an open set in Y. It follows from the lower hemicontinuity of $\Phi\colon X \to Y$ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap B_Y(\mathbf{q}, \varepsilon)$ is non-empty whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < \varepsilon$. In particular, given any positive integer s, there exists some positive integer δ_s such that, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_s$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < 1/s$.

Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exist positive integers $k(1), k(2), k(3), \ldots$, where

$$k(1) < k(2) < k(3) < \cdots$$

such that $|\mathbf{x}_j - \mathbf{p}| < \delta_s$ for all positive integers j satisfying $j \geq k(s)$. There then exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $|\mathbf{y}_j - \mathbf{q}| < 1/s$ for all positive integers j and s satisfying $k(s) \leq j < k(s+1)$. Then $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. We have thus shown that if $\Phi \colon X \to Y$ is lower hemicontinuous at the point $\mathbf{p}$, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence in X converging to the point $\mathbf{p}$, and if $\mathbf{q} \in \Phi(\mathbf{p})$, then there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integer j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Next suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is not lower hemicontinuous at $\mathbf{p}$. Then there exists an open set V in Y such that $\Phi(\mathbf{p}) \cap V$ is non-empty but there does not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{p} - \mathbf{x}| < \delta$. Let $\mathbf{q} \in \Phi(\mathbf{p})$. There then exists an infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap V = \emptyset$ for all positive integers j. It is not then possible to construct an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

The result follows. ∎

4.5. Intersections of Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \to Y$ be correspondences between X and Y. The *intersection* $\Phi \cap \Psi$ of the correspondences Φ and Ψ is defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$.

Proposition 4.21

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi\colon X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ be the correspondence defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

Proof

Let

$$W = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then W is the complement of the graph $\mathrm{Graph}(\Psi)$ of Ψ in $X \times Y$. The graph of Ψ is closed in $X \times Y$, by assumption. It follows that W is open in $X \times Y$.

Let $\mathbf{x} \in X$. The subset $\Psi(\mathbf{x})$ of Y is closed in Y, because the graph of the correspondence Ψ is closed. It follows from the compactness of $\Phi(\mathbf{x})$ that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ is a closed subset of the compact set $\Phi(\mathbf{x})$, and must therefore be compact. Thus the correspondence $\Phi \cap \Psi$ is compact-valued.

Now let $\mathbf{p}$ be a point of X, and let V be any open set in Y for which $\Phi(\mathbf{p}) \cap \Psi(\mathbf{p}) \subset V$. In order to prove that $\Phi \cap \Psi$ is upper hemicontinuous we must show that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \text{either } \mathbf{y} \in V \text{ or else } \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then U is the union of the subsets $X \times V$ and W of $X \times Y$, where both these subsets are open in $X \times Y$. It follows that U is open in $X \times Y$. Moreover if $\mathbf{y} \in \Phi(\mathbf{p})$ then either $\mathbf{y} \in \Phi(\mathbf{p}) \cap \Psi(\mathbf{p})$, in which case $\mathbf{y} \in V$, or else $\mathbf{y} \notin \Psi(\mathbf{p})$. It follows that $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Now it follows from Proposition 4.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. Therefore there exists some positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $(\mathbf{x}, \mathbf{y}) \in X \times Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. Now if $(\mathbf{x}, \mathbf{y})$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ then $(\mathbf{x}, \mathbf{y}) \in U$ but $(\mathbf{x}, \mathbf{y}) \notin W$. It follows from the definition of U that $\mathbf{y} \in V$. Thus $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

4.6. Berge's Maximum Theorem

Lemma 4.22

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi \colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let $f \colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < c \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) < c\}.$$

It follows from the continuity of the function f that U is open in $X \times Y$. It then follows from Proposition 4.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. The result follows. ∎

Lemma 4.23

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then

$$\{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

Proof

Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) > c\},$$

and let

$$W = \{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\},$$

Let $\mathbf{p} \in W$. Then there exists $\mathbf{y} \in \Phi(\mathbf{p})$ for which $(\mathbf{p}, \mathbf{y}) \in U$. There then exist subsets W_X of X and W_Y of Y, where W_X is open in X and W_Y is open in Y, such that $\mathbf{p} \in W_X$, $\mathbf{y} \in W_Y$ and $W_X \times W_Y \subset U$ (see Lemma 4.5). There then exists some positive real number δ_1 such that $\mathbf{x} \in W_X$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_1$.

Now $\Phi(\mathbf{p}) \cap W_Y \neq \emptyset$, because $\mathbf{y} \in \Phi(\mathbf{p}) \cap W_Y$. It follows from the lower hemicontinuity of the correspondence Φ that there exists some positive real number δ_2 such that $\Phi(\mathbf{x}) \cap W_Y \neq \emptyset$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then there exists $\mathbf{y} \in \Phi(\mathbf{x})$ for which $\mathbf{y} \in W_Y$. But then $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$ and therefore $(\mathbf{x}, \mathbf{y}) \in U$, and thus $f(\mathbf{x}, \mathbf{y}) > c$. The result follows. $\blacksquare$

Theorem 4.24 (Berge's Maximum Theorem)

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f: X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi: X \to Y$ is both upper hemicontinuous and lower hemicontinuous. Let the real-valued function $m: X \to \mathbb{R}$ be defined on X such that

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M: X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m: X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M: X \rightrightarrows Y$ is upper hemicontinuous.

Proof

Let $\mathbf{x} \in X$. Then $\Phi(\mathbf{x})$ is a non-empty compact subset of Y. It is thus a closed bounded subset of $\mathbb{R}^m$. It follows from the Extreme Value Theorem (Theorem 2.20) that there exists at least one point $\mathbf{y}^*$ of $\Phi(\mathbf{x})$ with the property that $f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Then $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^* \in M(\mathbf{x})$. Moreover

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}.$$

It follows from the continuity of f that the set $M(\mathbf{x})$ is closed in Y (see Corollary 2.18). It is thus a closed subset of the compact set $\Phi(\mathbf{x})$ and must therefore itself be compact.

Let some positive number ε be given. Then $f(\mathbf{p}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. It follows from Lemma 4.22 that

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X, and thus there exists some positive real number δ_1 such that $f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $\mathbf{y} \in \Phi(\mathbf{x})$ Then $m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$.

The correspondence $\Phi\colon X \rightrightarrows Y$ is also lower hemicontinuous. It therefore follows from Lemma 4.23 that there exists some positive real number δ_2 such that, given any $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_2$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ for which $f(\mathbf{x}, \mathbf{y}) > m(\mathbf{p}) - \varepsilon$. It follows that $m(\mathbf{x}) > m(\mathbf{p}) - \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$m(\mathbf{p}) - \varepsilon < m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $m\colon X \to \mathbb{R}$ is continuous on X.

It only remains to prove that the correspondence $M \colon X \rightrightarrows Y$ is upper hemicontinuous. Let

$$\Psi(\mathbf{x}) = \{\mathbf{y} \in Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then

$$\mathrm{Graph}(\Psi) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

Thus $\mathrm{Graph}(\Psi)$ is the preimage of zero under the continuous real-valued function that sends $(\mathbf{x}, \mathbf{y}) \in X \times Y$ to $f(\mathbf{x}, \mathbf{y}) - m(\mathbf{x})$. It follows that $\mathrm{Graph}(\Psi)$ is a closed subset of $X \times Y$.

Now $M(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ for all $\mathbf{x} \in X$, where the correspondence Φ is compact-valued and upper hemicontinuous and the correspondence Ψ has closed graph. It follows from Proposition 4.21 that the correspondence M must itself be both compact-valued and upper hemicontinuous. This completes the proof of Berge's Maximum Theorem. ∎