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4.3. Compact-Valued Upper Hemicontinuous Correspondences

Lemma 4.14

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Proof

Given any open set V in Y, let

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

It follows from the upper hemicontinuity of Φ that $\Phi^+(V)$ is open in X for all open sets V in Y (see Lemma 4.1). Now the empty set $\emptyset$ is open in Y. It follows that $\Phi^+(\emptyset)$ is open in X. But

$$\Phi^+(\emptyset) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset \emptyset\} = \{\mathbf{x} \in X : \Phi(\mathbf{x}) = \emptyset\}.$$

It follows that the set of point $\mathbf{x}$ in X for which $\Phi(\mathbf{x}) = \emptyset$ is open in X, and therefore the set of points $\mathbf{x} \in X$ for which $\Phi(\mathbf{x}) \neq \emptyset$ is closed in X, as required. ∎

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Given any subset S of X, we define the *image* $\Phi(S)$ of S under the correspondence Φ to be the subset of Y defined such that

$$\Phi(S) = \bigcup_{\mathbf{x} \in S} \Phi(\mathbf{x})$$

Lemma 4.15

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.

Proof

Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_{\mathbf{p}}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_{\mathbf{p}}$. Then $\Phi(\mathbf{p}) \subset U_{\mathbf{p}}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_{\mathbf{p}}$ in X such that $\Phi(\mathbf{x}) \subset U_{\mathbf{p}}$ for all $\mathbf{x} \in N_{\mathbf{p}}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_{\mathbf{p}}$ of open sets in Y covers $\Phi(N_{\mathbf{p}})$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows. ∎

Proposition 4.16

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \varepsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ε of some point of $\Phi(\mathbf{p})$.

Proof

Let $\Phi\colon X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi\colon X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$ (see Proposition 4.9). There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$ is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 4.17

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

Proof
Throughout this proof, let us say that the correspondence Φ satisfies the *constrained convergent subsequence criterion* if (and only if), given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$. We must prove that the correspondence $\Phi \colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion if and only if it is compact-valued and upper hemicontinuous.

Suppose first the the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Applying this criterion when $\mathbf{x}_j = \mathbf{p}$ for all positive integers j, we conclude that every infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\Phi(\mathbf{p})$ has a convergent subsequence, and therefore $\Phi(\mathbf{x})$ is compact.

Let

$$D = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}.$$

We show that D is closed in X. Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

be a sequence of points of D converging to some point of $\mathbf{p}$ of X. Then $\Phi(\mathbf{x}_j)$ is non-empty for all positive integers j, and therefore there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j. The constrained convergent subsequence criterion ensures that this infinite sequence in Y must have a subsequence that converges to some point of $\Phi(\mathbf{p})$. It follows that $\phi(\mathbf{p})$ is non-empty, and thus $\mathbf{p} \in D$.

Let $\mathbf{p}$ be a point of the complement of D. Then $\Phi(\mathbf{p}) = \emptyset$. There then exists $\delta > 0$ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\Phi(\mathbf{x}) \subset V$ for all open sets V in Y. It follows that the correspondence Φ is upper hemicontinuous at those points $\mathbf{p}$ for which $\Phi(\mathbf{p}) = \emptyset$.

Now consider the situation in which $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion and $\mathbf{p}$ is some point of X for which $\Phi(\mathbf{p}) \neq \emptyset$. Let $K = \Phi(\mathbf{p})$. Then K is a compact non-empty subset of Y. Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Suppose that there did not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It would then follow that there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $\mathbf{y}_j \notin V$.

Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, and thus the constrained convergent subsequence criterion satisfied by the correspondence Φ would ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. But then $\mathbf{q} \notin V$, because $\mathbf{y}_{k_j} \notin V$ for all positive integers j, and the complement $Y \setminus V$ of V is closed in Y. But $\Phi(\mathbf{p}) \subset V$, and $\mathbf{q} \in \Phi(\mathbf{p})$, and therefore $\mathbf{q} \in V$. Thus a contradiction would arise were there not to exist a positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus such a real number δ must exist, and thus the constrained convergent subsequence criterion ensures that the correspondence $\Phi \colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

It remains to show that any compact-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Let $\Phi\colon X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. It follows from Lemma 4.14 that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

be infinite sequences in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Then $\Phi(\mathbf{p})$ is non-empty, because

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X (see Lemma 4.14). Let $K = \Phi(\mathbf{p})$. Then K is compact, because $\Phi \colon X \rightrightarrows Y$ is compact-valued by assumption. For each integer j let $d(\mathbf{y}_j, K)$ denote the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K. There then exists an infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K such that $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$. for all positive integers j.

Now the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ ensures that $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. Indeed, given any positive real number ε, the set $B_Y(K, \varepsilon)$ of points of Y that lie within a distance ε of a point of K is an open set containing $\Phi(\mathbf{p})$. It follows from the upper hemicontinuity of Φ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(K, \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. But then $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and therefore $d(\mathbf{y}_j, K) < \varepsilon$ whenever $j \geq N$.

Now the compactness of K ensures that the infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K has a subsequence

$$\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \mathbf{z}_{k_3}, \ldots$$

that converges to some point $\mathbf{q}$ of K. Now $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$ for all positive integers j, and $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. It follows that $\mathbf{y}_{k_j} \to \mathbf{q}$ as $j \to +\infty$. Morever $\mathbf{q} \in \Phi(\mathbf{p})$. We have therefore verified that the constrained convergent subsequence criterion is satisfied by any correspondence $\Phi\colon X \rightrightarrows Y$ that is compact-valued and upper hemicontinuous. This completes the proof. ∎

Proposition 4.18

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof using Proposition 4.10

Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. If $\Phi(\mathbf{p}) = \emptyset$ then it follows from Lemma 4.14 that there exists some positive real number δ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

Suppose that $\Phi(\mathbf{p}) \neq 0$. Let $K = \Phi(\mathbf{p})$. Then K is a compact subset of Y, because the correspondence Φ is compact-valued. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$. It follows from Proposition 4.10 that there exists some positive real number δ_1 such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $d_Y(\mathbf{y}, K) < \delta_1$, where

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}.$$

Let

$$V = \{\mathbf{y} \in Y : d_Y(\mathbf{y}, K) < \delta_1\}.$$

Then V is open in Y because the function sending $\mathbf{y} \in Y$ to $d(\mathbf{y}, K)$ is continuous on Y (see Lemma 4.8). Also $\Phi(\mathbf{p}) \subset V$. It follows from the upper hemicontinuity of the correspondence Φ that there exists some positive number δ_2 such that $\Phi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\Phi(\mathbf{x}) \subset V$. But then $d(\mathbf{y}, K) < \delta_1$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Moreover $|\mathbf{x} - \mathbf{p}| < \delta_1$. It follows that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$, and therefore $\mathbf{x} \in W$. This shows that W is an open subset of X, as required. ∎

Proof using Proposition 4.17

Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. Suppose that there did not exist any strictly positive real number δ with the property that $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any positive real number δ, there would exist points $\mathbf{x}$ of X and $\mathbf{y}$ of Y such that $|\mathbf{x} - \mathbf{p}| < \delta$, $\mathbf{y} \in \Phi(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \notin U$. Therefore there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively such that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ and $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$ for all positive integers j.

The correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous. Proposition 4.17 would therefore ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of Y converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. Now the complement of U in $X \times Y$ is closed in $X \times Y$, because U is open in $X \times Y$ and $(\mathbf{x}_j, \mathbf{y}_j) \not\in U$. It would therefore follow that $(\mathbf{p}, \mathbf{q}) \not\in U$ (see Proposition 4.6). But this gives rise to a contradiction, because $\mathbf{q} \in \Phi(\mathbf{p})$ and $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. In order to avoid the contradiction, there must exist some positive real number δ with the property that with the property that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. The result follows. ∎

Proof using Compactness (Heine-Borel) directly

Let $\Phi\colon X \to Y$ be a compact-valued upper hemicontinuous correspondence, and let U be a subset of $X \times Y$ that is open in $X \times Y$. Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

We must prove that W is open in X.

Let $K = \Phi(\mathbf{p})$. Then, given any point $\mathbf{y}$ of K, there exists an open set $M_{\mathbf{p},\mathbf{y}}$ in X and an open set $V_{\mathbf{p},\mathbf{y}}$ in Y such that $M_{\mathbf{p},\mathbf{y}} \times V_{\mathbf{p},\mathbf{y}} \subset U$ (see Lemma 4.5). Now every open cover of K has a finite subcover, because K is compact. Therefore there exist points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ of K such that

$$K \subset V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Let

$$M_{\mathbf{p}} = M_{\mathbf{p},\mathbf{y}_1} \cap M_{\mathbf{p},\mathbf{y}_2} \cap \cdots \cap M_{\mathbf{p},\mathbf{y}_k}$$

and

$$V_{\mathbf{p}} = V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Then

$$M_{\mathbf{p}} \times V_{\mathbf{p}} \subset \bigcup_{j=1}^{k} (M_{\mathbf{p}} \times V_{\mathbf{p},\mathbf{y}_j}) \subset \bigcup_{j=1}^{k} (M_{\mathbf{p},\mathbf{y}_j} \times V_{\mathbf{p},\mathbf{y}_j}) \subset U.$$

Now $M_{\mathbf{p}}$ is open in X, because it is the intersection of a finite number of subsets of X that are open in X. Also it follows from the upper hemicontinuity of the correspondence Φ that $\Phi^{+}(V_{\mathbf{p}})$ is open in X, where

$$\Phi^{+}(V_{\mathbf{p}}) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V_{\mathbf{p}}\}$$

(see Lemma 4.1). Let $N_{\mathbf{p}} = M_{\mathbf{p}} \cap \Phi^{+}(V_{\mathbf{p}})$. Then $N_{\mathbf{p}}$ is open in X and $\mathbf{p} \in N_{\mathbf{p}}$. Now if $\mathbf{x} \in N_{\mathbf{p}}$ then $\mathbf{x} \in M_{\mathbf{p}}$ and $\Phi(\mathbf{x}) \subset V_{\mathbf{p}}$, and therefore $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. We have thus shown that $N_{\mathbf{p}} \subset W$ for all $\mathbf{p} \in W$, where

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Thus W is the union of the subsets $N_{\mathbf{p}}$ as $\mathbf{p}$ ranges over the points of W. Moreover the set $N_{\mathbf{p}}$ is open in X for each $\mathbf{p} \in W$. It follows that W must itself be open in X. Indeed, given any point $\mathbf{p}$ of W, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N_{\mathbf{p}} \subset W.$$

The result follows. $\blacksquare$

Remark

The various proofs of Proposition 4.18 were presented in the contexts of correspondences between subsets of Eucldean spaces. All these proofs generalize easily so as to apply to correspondence between subsets of metric spaces. The last of the proofs can be generalized without difficulty so as to apply to correspondences between topological spaces. Inded the notion of *correspondences* between topological spaces is defined so that a correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y associates to each point of X a subset $\Phi(\mathbf{x})$ of Y. Such a correspondence is said to be upper hemicontinuous at a point p of X if, given any open subset V of Y for which $\Phi(p) \subset V$, there exists an open set $N(p)$ in X such that $\Phi(x) \subset V$ for all $x \in N$.

The last of the proofs of Proposition 4.18 presented above can be generalized to show that, given a compact-valued correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y, and given a subset U of Y, the set

$$\{x \in X : (x, y) \in U \text{ for all } y \in \Phi(x)\}$$

is open in X.

Remark

It should be noted that other results proved in this section do not necessarily generalize to correspondences $\Phi\colon X \rightrightarrows Y$ mapping the topological space X into an arbitrary topological space Y. For example all closed-valued upper hemicontinuous correspondences between metric spaces have closed graphs. The appropriate generalization of this result states that any closed-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ from a topological space X to a regular topological space Y has a closed graph. To interpret this, one needs to know the definition of what is meant by saying that a topological space is *regular*. A topological space Y is said to be *regular* if, given any closed subset F of Y, and given any point p of the complement $Y \setminus F$ of F, there exist open sets V and W in Y such that $F \subset V$, $p \in W$ and $V \cap W = \emptyset$. Metric spaces are regular. Also compact Hausdorff spaces are regular.