MA3486 Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 12 (February 12, 2016)

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Lemma 4.8

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let S be a non-empty subset of X, and let

$$d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$$

for all $\mathbf{x} \in X$. Then the function sending $\mathbf{x}$ to $d(\mathbf{x}, S)$ for all $\mathbf{x} \in X$ is a continuous function on X.

Proof

Let $f(\mathbf{x}) = d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$ for all $\mathbf{x} \in X$.

Let $\mathbf{x}$ and $\mathbf{x}'$ be points of X. It follows from the Triangle Inequality that

$$f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{s}| \leq |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{s}|$$

for all $\mathbf{s} \in S$, and therefore

$$|\mathbf{x}' - \mathbf{s}| \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{s} \in S$. Thus $f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$ is a lower bound for the quantities $|\mathbf{x}' - \mathbf{s}|$ as $\mathbf{s}$ ranges over the set S, and therefore cannot exceed the greatest lower bound of these quantities.

It follows that

$$f(\mathbf{x}') = \inf\{|\mathbf{x}' - \mathbf{s}| : \mathbf{s} \in S\} \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|,$$

and thus

$$f(\mathbf{x}) - f(\mathbf{x}') \leq |\mathbf{x} - \mathbf{x}'|.$$

Interchanging $\mathbf{x}$ and $\mathbf{x}'$, it follows that

$$f(\mathbf{x}') - f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}'|.$$

Thus

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{x}, \mathbf{x}' \in X$. It follows that the function $f\colon X \to \mathbb{R}$ is continuous, as required. ∎

The multidimensional Heine-Borel Theorem (Theorem 3.3) ensures that a subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proposition 4.9

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbb{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ε with the property that $B_X(K, \varepsilon) \subset V$, where $B_X(K, \varepsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ε of some point of K.

Proof using the Bolzano-Weierstrass Theorem

Suppose that the proposition were false. Then there would exist infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots$ such that $\mathbf{x}_j \in K$, $\mathbf{w}_j \in X \setminus V$ and $|\mathbf{w}_j - \mathbf{x}_j| < 1/j$ for all positive integers j. The set K is both closed and bounded in $\mathbb{R}^n$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) would then ensure the existence of a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to some point $\mathbf{q}$ of K. Moreover $\lim\limits_{j \to +\infty} (\mathbf{w}_j - \mathbf{x}_j) = \mathbf{0}$, and therefore

$$\lim_{j \to \infty} \mathbf{w}_{k_j} = \lim_{j \to \infty} \mathbf{x}_{k_j} = \mathbf{q}.$$

But $\mathbf{w}_j \in X \setminus V$. Moreover $X \setminus V$ is closed in X, and therefore any sequence of points in $X \setminus V$ that converges in X must converge to a point of $X \setminus V$ (see Lemma 2.16). It would therefore follow that $\mathbf{q} \in K \cap (X \setminus V)$. But this is impossible, because $K \subset V$ and therefore $K \cap (X \setminus V) = \emptyset$. Thus a contradiction would follow were the proposition false. The result follows. ■

Proof using the Heine-Borel Theorem

It follows from the multidimensional Heine-Borel Theorem (Theorem 3.3) that the set K is compact, and thus every open cover of K has a finite subcover. Given point $\mathbf{x}$ of K let $\varepsilon_{\mathbf{x}}$ be a positive real number with the property that

$$B_X(\mathbf{x}, 2\varepsilon_{\mathbf{x}}) \subset V,$$

where

$$B_X(\mathbf{x}, r) = \{\mathbf{x}' \in X : |\mathbf{x}' - \mathbf{x}| < r\}$$

for all positive integers r. The collection of open balls $B_X(\mathbf{x}, \varepsilon_{\mathbf{x}})$ determined by the points $\mathbf{x}$ of K covers K. By compactness this open cover of K has a finite subcover. Therefore there exist points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of K such that

$$K \subset B(\mathbf{x}_1, \varepsilon_{\mathbf{x}_1}) \cup B(\mathbf{x}_2, \varepsilon_{\mathbf{x}_2}) \cup \cdots \cup B(\mathbf{x}_k, \varepsilon_{\mathbf{x}_k}).$$

Let ε be the minimum of $\varepsilon_{\mathbf{x}_1}, \varepsilon_{\mathbf{x}_2}, \ldots, \varepsilon_{\mathbf{x}_k}$. If $\mathbf{x}$ is a point of K then $\mathbf{x} \in B_X(\mathbf{x}_j, \varepsilon_{\mathbf{x}_j}$ for some integer j between 1 and k. But it then follows from the Triangle Inequality that

$$B(\mathbf{x}, \varepsilon) \subset B_X(\mathbf{x}_j, 2\varepsilon_{\mathbf{x}_j}) \subset V.$$

It follows from this that

$$B_X(K, \varepsilon) \subset V,$$

as required. ∎

Proof using the Extreme Value Theorem
Let $f\colon K \to \mathbb{R}$ be defined such that

$$f(\mathbf{x}) = \inf\{|\mathbf{z} - \mathbf{x}| : \mathbf{z} \in X \setminus V\}.$$

for all $\mathbf{x} \in K$. It follows from Lemma 4.8 that the function f is continuous on K.

Now $K \subset V$ and therefore, given any point $\mathbf{x} \in K$, there exists some positive real number δ such that the open ball of radius δ about the point $\mathbf{x}$ is contained in V, and therefore $f(\mathbf{x}) \geq \delta$. It follows that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$.

It follows from the Extreme Value Theorem for continuous real-valued functions on closed bounded subsets of Euclidean spaces (Theorem 2.20) that the function $f\colon K \to \mathbb{R}$ attains its minimum value at some point of K. Let that minimum value be ε. Then $f(\mathbf{x}) \geq \varepsilon > 0$ for all $\mathbf{x} \in K$, and therefore $|\mathbf{x} - \mathbf{x}| \geq \varepsilon > 0$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in X \setminus V$. It follows that $B_X(K, \varepsilon) \subset V$, as required. ∎

Example

Let

$$F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } xy \geq 1\}.$$

and let

$$V = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Note that if $(x, y) \in F$ then $x > 0$ and $y > 0$, because $xy = 1$. It follows that $F \subset V$. Also F is a closed set in $\mathbb{R}^2$ and V is an open set in $\mathbb{R}^2$. However F is not a compact subset of $\mathbb{R}^2$ because F is not bounded.

We now show that there does not exist any positive real number ε with the property that $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$, where $B_{\mathbb{R}^2}(F, \varepsilon)$ denotes the set of points of $\mathbb{R}^2$ that lie within a distance ε of some point of F. Indeed let some positive real number ε be given, let x be a positive real number satisfying $x > 2\varepsilon^{-1}$, and let $y = x^{-1} - \frac{1}{2}\varepsilon$. Then $y < 0$, and therefore $(x, y) \notin V$. But $(x, y + \frac{1}{2}\varepsilon) \in F$, and therefore $(x, y) \in B_{\mathbb{R}^2}(F, \varepsilon)$. This shows that there does not exist any positive real number ε for which $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$.

Proposition 4.10

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let K be a non-empty compact subset of Y, and let U be an subset in $X \times Y$ that is open in $X \times Y$. Let

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}$$

for all $\mathbf{y} \in Y$. Let $\mathbf{p}$ be a point of X with the property that $(\mathbf{p}, \mathbf{z}) \in U$ for all $\mathbf{z} \in K$. Then there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $d(\mathbf{y}, K) < \delta$.

Proof
Let
$$\tilde{K} = \{(\mathbf{p}, \mathbf{z}) : \mathbf{z} \in K\}.$$
Then $\tilde{K}$ is a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$. It follows from Proposition 4.9 that there exists some positive real number ε such that
$$B_{X\times Y}(\tilde{K}, \varepsilon) \subset U$$
where $B_{X\times Y}(\tilde{K}, \varepsilon)$ denotes that subset of $X \times Y$ consisting of those points $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ that lie within a distance ε of a point of $\tilde{K}$. Now a point $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ belongs to $B_{X\times Y}(\tilde{K}, \varepsilon)$ if and only if there exists some point $\mathbf{z}$ of K for which
$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < \varepsilon^2.$$

Let $\delta = \varepsilon/\sqrt{2}$. If $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $d_Y(\mathbf{y}, K) < \delta$ then there exists some point $\mathbf{z}$ of K for which $|\mathbf{y} - \mathbf{z}| < \delta$. But then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < 2\delta^2 = \varepsilon^2,$$

and therefore $(\mathbf{x}, \mathbf{y}) \in U$, as required. $\blacksquare$

Proposition 4.11

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph $\mathrm{Graph}(\Phi)$ of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.

Proof

Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$.

4. Correspondences and Hemicontinuity (continued)

Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$. Also $\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required. $\blacksquare$

Proposition 4.12

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof using Proposition 4.10

Let $\mathbf{p}$ be a point of X, let V be an open set satisfying $\Phi(\mathbf{p}) \subset V$, and let $K = Y \setminus V$. The compact set Y is closed and bounded in $\mathbb{R}^m$. Also K is closed in Y. It follows that K is a closed bounded subset of $\mathbb{R}^m$ (see Lemma 2.23). Let U be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then U is open in $X \times Y$, because $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$, because $\Phi(\mathbf{p}) \cap K = \emptyset$. It follows from Proposition 4.10 that there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in K$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\mathbf{y} \notin \Phi(\mathbf{x})$ for all $\mathbf{y} \in K$, and therefore $\Phi(\mathbf{x}) \subset V$, where $V = Y \setminus K$. Thus the correspondence Φ is upper hemicontinuous at $\mathbf{p}$, as required. ∎

Proof using the Bolzano-Weierstrass Theorem

Let V be a subset of Y that is open in Y, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \subset V$. Let $F = Y \setminus V$. Then the set F is a subset of Y that is closed in Y, and $\Phi(\mathbf{p}) \cap F = \emptyset$. Now Y is a closed bounded subset of $\mathbb{R}^m$, because it is compact (Theorem 3.3). It follows that F is closed in $\mathbb{R}^m$ (Lemma 2.23).

Suppose that there did not exist any positive number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ for all positive integers j. There would then exist an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ of elements of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j) \cap F$ for all positive integers j. Then $(\mathbf{x}_j, \mathbf{y}_j) \in \mathrm{Graph}(\Phi)$ for all positive integers j. Moreover the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ would be bounded, because the set Y is bounded.

It would therefore follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there would exist a convergent subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of the sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$. Let $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}_{k_j}$. Then $\mathbf{q} \in F$, because the set F is closed in Y and $\mathbf{y}_{k_j} \in F$ for all positive integers j (see Lemma 2.16). Similarly $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, because the set $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$, $(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) \in \mathrm{Graph}(\Phi)$ for all positive integers j, and

$$(\mathbf{p}, \mathbf{q}) = \lim_{j \to +\infty} (\mathbf{x}_{k_j}, \mathbf{y}_{k_j}).$$

But were there to exist $(\mathbf{p}, \mathbf{q}) \in X \times Y$ for which $\mathbf{q} \in F$ and $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, it would follow that $\mathbf{q} \in \Phi(\mathbf{p}) \cap F$. But this is impossible, because $\Phi(\mathbf{p}) \cap F = \emptyset$. Thus a contradiction would arise were there to exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X for which $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ and $\displaystyle\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Therefore no such infinite sequence can exist, and therefore there must exist some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. We conclude that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}$$

is open in X. The result follows. ∎

Corollary 4.13

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the function $\varphi\colon X \to Y$ is continuous.

Proof

Let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\varphi)$, and therefore $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. The subset Y of $\mathbb{R}^m$ is compact. It therefore follows from Proposition 4.12 that the correspondence Φ is upper hemicontinuous. It then follows from Lemma 4.3 that the function $\varphi\colon X \to Y$ is continuous, as required. ∎