MA3486 Fixed Point Theorems and Economic Equilibria School of Mathematics, Trinity College Hilary Term 2016 Lecture 11 (February 11, 2016)

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4.2. The Graph of a Correspondence

Let *m* and *n* be integers. Then the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ of the Euclidean spaces \mathbb{R}^n and \mathbb{R}^m of dimensions *n* and *m* is itself a Euclidean space of dimension n + m whose Euclidean norm is characterized by the property that

$$|(\mathbf{x}, \mathbf{y})|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Lemma 4.4

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ be infinite sequences of points in \mathbb{R}^n and \mathbb{R}^m respectively, and let $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$. Then the infinite sequence

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point (\mathbf{p}, \mathbf{q}) if and only if the infinite sequence Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point \mathbf{p} and the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converges to the point \mathbf{q} .

Proof

Suppose that the infinite sequence

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point (\mathbf{p}, \mathbf{q}) . Let some strictly positive real number ε be given. Then there exists some positive integer N such that

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \ge N$. But then

$$|\mathbf{x}_j - \mathbf{p}| < arepsilon$$
 and $|\mathbf{y}_j - \mathbf{q}| < arepsilon$

whenever $j \ge N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$.

Conversely suppose that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/\sqrt{2}$ whenever $j \ge N_1$ and $|\mathbf{y}_j - \mathbf{q}| < \varepsilon/\sqrt{2}$ whenever $j \ge N_2$. Let N be the maximum of N_1 and N_2 . Then

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \ge N$. It follows that $(\mathbf{x}_j, \mathbf{y}_j) \to (\mathbf{p}, \mathbf{q})$ as $j \to +\infty$, as required.

Lemma 4.5

Let X and Y be subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and let V be a subset of $X \times Y$. Then V is open in $X \times Y$ if and only if, given any point (\mathbf{p}, \mathbf{q}) of V, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$, there exist subsets W_X and W_Y of X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$, W_X is open in X, W_Y is open in Y and $W_X \times W_Y \subset V$.

Proof

Let V be a subset of $X \times Y$ and let $(\mathbf{p}, \mathbf{q}) \in V$, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$.

Suppose that V is open in $X \times Y$. Then there exists a positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in V$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < \delta^2.$$

4. Correspondences and Hemicontinuity (continued)

Let

$$W_X = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < rac{\delta}{\sqrt{2}}
ight\}$$

and

$$W_Y = \left\{ \mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| < rac{\delta}{\sqrt{2}}
ight\}$$

If $\mathbf{x} \in W_X$ and $\mathbf{y} \in W_Y$ then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < 2\left(\frac{\delta}{\sqrt{2}}\right)^2 = \delta^2$$

and therefore $(\mathbf{x}, \mathbf{y}) \in V$. It follows that $W_X \times W_Y \subset V$.

Conversely suppose that there exist open sets W_X and W_Y in Xand Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$ and $W_X \times W_Y \subset V$. Then there exists some positive real number δ such that $\mathbf{x} \in W_X$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and also $\mathbf{y} \in W_Y$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \delta$. If (\mathbf{x}, \mathbf{y}) is a point of $X \times Y$ that lies within a distance δ of (\mathbf{p}, \mathbf{q}) then $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{y} - \mathbf{q}| < \delta$, and therefore $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$. But $W_X \times W_Y \subset V$. It follows that the open ball of radius δ about the point (\mathbf{p}, \mathbf{q}) is wholly contained within the subset V of $X \times Y$. The result follows.

Proposition 4.6

Let X and Y be subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if

 $(\lim_{j \to \infty} \mathbf{x}_j, \lim_{j \to \infty} \mathbf{y}_j) \in G$

for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.

Proof

Suppose that G is closed in $X \times Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be an infinite sequence in X converging to some point \mathbf{p} of X and let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ be an infinite sequence in Y converging to a point \mathbf{q} of Y, where $(\mathbf{x}_j, \mathbf{x}_j) \in G$ for all positive integers j. We must prove that $(\mathbf{p}, \mathbf{q}) \in G$. Now the infinite sequence consisting of the ordered pairs $(\mathbf{x}_j, \mathbf{y}_j)$ converges in $X \times Y$ to (\mathbf{p}, \mathbf{q}) (see Lemma 4.4). Now every infinite sequence contained in G that converges to a point of $X \times Y$ must converge to a point of G, because G is closed in $X \times Y$ (see Lemma 2.16). It follows that $(\mathbf{p}, \mathbf{q}) \in G$. Now suppose that *G* is not closed in $X \times Y$. Then the complement of *G* in $X \times Y$ is not open, and therefore there exists a point (\mathbf{p}, \mathbf{q}) of $X \times Y$ that does not belong to *G* though every open ball of positive radius about the point (\mathbf{p}, \mathbf{q}) intersects *G*. It follows that, given any positive integer *j*, the open ball of radius 1/j about the point (\mathbf{p}, \mathbf{q}) intersects *G* and therefore there exist $\mathbf{x}_j \in X$ and $\mathbf{y}_j \in Y$ for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $|\mathbf{y}_j - \mathbf{q}| < 1/j$ and $(\mathbf{x}_j, \mathbf{y}_j) \in G$. Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ and $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$ and therefore

 $(\lim_{j\to\infty} \mathbf{x}_j, \lim_{j\to\infty} \mathbf{y}_j) \notin G.$

The result follows.

Definition

Let X and Y be subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and let $\varphi \colon X \to Y$ be a function from X and Y. The graph $\operatorname{Graph}(\varphi)$ of the function φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$Graph(\varphi) = \{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} = \varphi(\mathbf{x}) \}.$$

Definition

Let X and Y be subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and let $\Phi: X \Rightarrow Y$ be a correspondence between X and Y. The graph $\operatorname{Graph}(\Phi)$ of the correspondence Φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\operatorname{Graph}(\Phi) = \{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} \in \Phi(\mathbf{x}) \}.$$

Lemma 4.7

Let X and Y be subsets of \mathbb{R}^n and \mathbb{R}^m respectively, and let $\varphi \colon X \to Y$ be a function from X to Y. Suppose that $\varphi \colon X \to Y$ is continuous. Then the graph $\operatorname{Graph}(\varphi)$ of the function φ is closed in $X \times Y$.

Proof

Let $\psi \colon X \times Y \to Y$ be the function defined such that

$$\psi(\mathbf{x},\mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then $\operatorname{Graph}(\varphi) = \psi^{-1}(\{\mathbf{0}\})$, and $\{\mathbf{0}\}$ is closed in \mathbb{R}^m . It follows that $\operatorname{Graph}(\varphi)$ is closed in $X \times Y$ (see Corollary 2.18).

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \le 0. \end{cases}$$

Then the graph Graph(f) of the function f satisfies $Graph(f) = Z \cup H$, where

$$Z=\{(x,y)\in \mathbb{R}^2:x\leq 0 ext{ and } y=0\}$$

and

$$H = \{(x, y) \in \mathbb{R}^2 : x \ge 0 \text{ and } xy = 1\}.$$

Each of the sets Z and H is a closed set in \mathbb{R}^2 . It follows that $\operatorname{Graph}(f)$ is a closed set in \mathbb{R}^2 . However the function $f : \mathbb{R} \to \mathbb{R}$ is not continuous at 0.