MA3486 Fixed Point Theorems and Economic Equilibria
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2.10. Homeomorphisms between Subsets of Euclidean Spaces

Lemma 2.23

Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.

Proof

Let F be a subset of X. Then F is closed in X if and only if, given any point $\mathbf{p}$ of X for which $\mathbf{p} \notin F$, there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. It follows easily from this that is F is closed in $\mathbb{R}^n$ then F is closed in X.

Conversely suppose that F is closed in X, where X itself is closed in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that satisfies $\mathbf{p} \notin F$. Then either $\mathbf{p} \in X$ or $\mathbf{p} \notin X$.

Suppose that $\mathbf{p} \in X$. Then there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ.

Otherwise $\mathbf{p} \notin X$. Then there exists some strictly positive real number δ such that there is no point of X whose distance from the point $\mathbf{p}$ is less than δ, because X is closed in $\mathbb{R}^n$. But $F \subset X$. It follows that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. We conclude that the set F is closed in $\mathbb{R}^n$, as required. $\blacksquare$

Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. A function $\psi\colon Y \to X$ is the inverse of $\varphi\colon X \to Y$ if and only if $\psi(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in X$ and $\varphi(\psi(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in Y$. The function X is *bijective* if and only if it has a well-defined inverse $\psi\colon Y \to X$.

Definition

Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A function $\varphi\colon X \to Y$ is said to be a *homeomorphism* if and only if it is bijective and both $\varphi\colon X \to Y$ itself and its inverse are continuous functions.

Proposition 2.24

Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous bijective function from X to Y. Suppose that X is closed and bounded. Then $\varphi\colon X \to Y$ is a homeomorphism.

Proof

Let $\varphi\colon X \to Y$ be a continuous bijective function from X to Y, and let $\psi\colon Y \to X$ be the inverse of φ. Then $\varphi\colon X \to Y$ establishes a one-to-one correspondence between points of X and points of Y: given any point $\mathbf{x}$ of X, the point $\varphi(\mathbf{x})$ is the unique point of Y that corresponds to $\mathbf{x}$; given any point $\mathbf{y}$ of Y, the point $\psi(\mathbf{y})$ is the unique point of X that corresponds to $\mathbf{y}$.

In order to prove that the continuous bijective function $\varphi\colon X \to Y$ is a homeomorphism, we need to prove that its inverse $\psi\colon Y \to X$ is continuous. Let W be an open set in X. We must prove that its preimage $\psi^{-1}(W)$ is open in W. Let $F = X \setminus W$. Then F is closed in X, and X itself is closed in $\mathbb{R}^n$. It follows that F is closed in $\mathbb{R}^n$ (see Lemma 2.23). Also F is bounded, because X is bounded.

Now continuous functions between subsets of Euclidean spaces map closed bounded sets to closed bounded sets (see Proposition 2.21). It follows that $\varphi(F)$ is a closed subset of $\mathbb{R}^m$ and is thus closed in Y, and therefore its complement $Y \setminus \varphi(F)$ is open in Y.

But $Y \setminus \varphi(F) = \psi^{-1}(V)$. Indeed let $\mathbf{y} \in Y$. Then

$$\begin{aligned} & \mathbf{y} \in Y \setminus \varphi(F) \\ \iff \quad & \mathbf{y} \notin \varphi(F) \\ \iff \quad & \psi(\mathbf{y}) \notin F \\ \iff \quad & \psi(\mathbf{y}) \in V \\ \iff \quad & \mathbf{y} \in \psi^{-1}(V). \end{aligned}$$

It follows that $\psi^{-1}(V)$ is open in Y. We have shown that the preimage under ψ of every subset of X open in X is open in Y. It follows that $\psi \colon Y \to X$ is continuous (see Proposition 2.17). We conclude that $\varphi \colon X \to Y$ is a homeomorphism, as required. ∎

Example

A regular dodecahedron is a regular convex polyhedron in 3-dimensional Euclidean space with twelve faces that are regular pentagons. Let the closed bounded subset X of $\mathbb{R}^3$ be the surface of a regular dodecahedron centred on the origin. Then every straight ray with an endpoint at the origin will cut X in exactly one point. Let S^2 be the unit sphere in $\mathbb{R}^3$, so that

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and let $\varphi\colon X \to S^2$ be defined so that

$$\varphi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$$

for all $\mathbf{x} \in X$. Then $\varphi\colon X \to S^2$ is a continuous bijective function from X to S^2. Let $\psi S^2 \to X$ be the inverse of φ. It follows from Proposition 2.24 That $\varphi\colon X \to S^2$ is a homeomorphism, and therefore the map $\psi\colon S^2 \to X$ is continuous.

It might be instructive to ponder how one might set about constructing a proof that $\psi\colon S^2 \to X$ is continuous, using directly the "ε–δ" definition of continuity. Presumably one would first have to come up with algebraic expressions that specify what the map is, and presumably there would need to be twelve such algebraic expressions, each specifying map ψ over some portion of the unit sphere that gets mapped onto a pentagonal face of the dodecahedron. And moreover the regions over which these algebraic expressions apply would probably need to be specified using appropriate inequalities satisfied by appropriate angles that arise from some curvilinear coordinate system on the sphere.

3. Open Covers, Lebesgue Numbers and Compactness

3.1. Lebesgue Numbers

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 3.1

Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof

Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then

$$B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for all positive integers j, the open ball

$$B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$.

The sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there would exist a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots.$$

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 2.16). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. $\blacksquare$

Definition

Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 3.1 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition

The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\operatorname{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\mathrm{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

A *hypercube* in n-dimensional Euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq u_i + l\},$$

where l is a positive constant that is the length of the edges of the hypercube and $(u_1, u_2, \ldots, u_n)$ is the point in $\mathbb{R}^n$ at which the Cartesian coordinates of points in the hypercube attain their minimum values. The diameter of a hypercube with edges of length l is $l\sqrt{n}$.

Lemma 3.2

Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the $\mathrm{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof

The set X is bounded, and therefore there exists some positive real number M such that that if $(x_1, x_2, \ldots, x_n) \in X$ then $-M \leq x_j \leq M$ for $j = 1, 2, \ldots, n$. Choose k large enough to ensure that $2M/k < \delta_L/\sqrt{n}$. Then the large hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -M \leq x_j \leq M \text{ for } j = 1, 2, \ldots, n\}$$

can be subdivided into k^n hypercubes with edges of length l, where $l = 2M/k$.

Each of the smaller hypercubes is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_j \leq x_j \leq u_j + l \text{ for } j = 1, 2, \ldots, n\},$$

where $(u_1, u_2, \ldots, u_n)$ is the corner of the hypercube at which the Cartesian coordinates have their minimum values. If $\mathbf{p}$ is a point belonging to such a small hypercube, then all points of the hypercube lie within a distance $l\sqrt{n}$ of the point $\mathbf{p}$. It follows that the small hypercube is wholly contained within the open ball $B_{\mathbb{R}^n}(\mathbf{p}, \delta_L)$ of radius δ about the point $\mathbf{p}$.

Now the number of small hypercubes resulting from the subdivision is finite. Let $H_1, H_2, \ldots, H_s$ be a listing of the small hypercubes that intersect the set X, and let $A_i = H_i \cap X$. Then $\mathrm{diam}(H_i) \leq \sqrt{n}l < \delta_L$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

as required. ∎

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition

A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 3.3

(The Multidimensional Heine-Borel Theorem) A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Proof

Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j} \right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 3.1 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 3.2 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\mathrm{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. $\blacksquare$