MA3486 Fixed Point Theorems and Economic Equilibria
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2.8. The Multidimensional Extreme Value Theorem

Theorem 2.20 (The Multidimensional Extreme Value Theorem)

Let X be a closed bounded set in n-dimensional Euclidean space, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof

We prove the result for an arbitrary continuous real-valued function $f: X \to \mathbb{R}$ by showing that the result holds for a related continuous function $g: X \to \mathbb{R}$ that is known to be bounded above and below on X. Let $h: \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. Then the continuous function $h: \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 < h(t) < 1$ for all $t \in \mathbb{R}$.

Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on the closed bounded set X, and let $g\colon X \to \mathbb{R}$ be the continuous real-valued function defined on X such that

$$g(\mathbf{x}) = h(f(\mathbf{x})) = \frac{f(\mathbf{x})}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Then $-1 < g(\mathbf{x}) < 1$ for all $\mathbf{x} \in X$. The set of values of the function g is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then, for each positive integer j, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function g, and therefore there exists some point $\mathbf{x}_j$ in the set X for which $M - j^{-1} < g(\mathbf{x}_j) \leq M$. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is then a bounded sequence of points in $\mathbb{R}^m$, because the set X is bounded. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{v}$ of $\mathbb{R}^n$. Moreover this point $\mathbf{v}$ belongs to the set X because X is closed (see Lemma 2.16).

Now

$$M - \frac{1}{k_j} < g(\mathbf{x}_{k_j}) \leq M$$

for all positive integers j, and therefore $g(\mathbf{x}_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 2.7 that

$$g(\mathbf{v}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = M.$$

But $g(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. It follows that $h(f(\mathbf{x})) = g(\mathbf{x}) \leq g(\mathbf{v}) = h(f(\mathbf{v}))$ for all $\mathbf{x} \in X$. Moreover $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

On applying this result with the continuous function f replaced by the function $-f$, we conclude also that there exists some point $\mathbf{u}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$. The result follows. ∎

Proposition 2.21

Let X be a closed bounded set in $\mathbb{R}^n$ and let $\varphi\colon X \to \mathbb{R}^m$ be a continuous function. Then $\varphi(X)$ is a closed bounded set in $\mathbb{R}^m$.

Proof

Let $g\colon X \to \mathbb{R}$ be the real-valued function on X defined such that $g(\mathbf{x}) = |\varphi(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the function $g\colon X \to \mathbb{R}$ is continuous (because it is a composition of continuous functions). It follows from the Extreme Value Theorem (Theorem 2.20) that there exists some point $\mathbf{v}$ of X such that $g(\mathbf{x}) \leq g(\mathbf{v})$ for all $\mathbf{x} \in X$. Let $M = g(\mathbf{v})$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. Thus the set $\varphi(X)$ is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^m \setminus \varphi(X)$ and let $h\colon X \to \mathbb{R}^m$ be the real-valued function on X defined such that $h(\mathbf{x}) = |\varphi(\mathbf{x}) - \mathbf{q}|$ for all $\mathbf{x} \in X$. Then the function $h\colon X \to \mathbb{R}$ is continuous. It follows from the Extreme Value Theorem (Theorem 2.20) that there exists some point $\mathbf{u}$ of X such that $h(\mathbf{x}) \geq h(\mathbf{u})$ for all $\mathbf{x} \in X$. Let $\delta = h(\mathbf{u})$. Then $|\varphi(\mathbf{x}) - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$, and thus the open ball in $\mathbb{R}^m$ of radius δ about the point $\mathbf{q}$ is contained in the complement $\mathbb{R}^m \setminus \varphi(X)$ of $\varphi(X)$. It follows that $\mathbb{R}^m \setminus \varphi(X)$ is open in $\mathbb{R}^m$, and thus the set $\varphi(X)$ is closed in $\mathbb{R}^m$. Thus $\varphi(X)$ is both closed and bounded, as required. ∎

2.9. Uniform Continuity for Functions of Several Real Variables

Definition

Let X be a subset of $\mathbb{R}^n$. A function $\varphi\colon X \to \mathbb{R}^m$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 2.22

Let X be a subset of $\mathbb{R}^n$ that is both closed and bounded. Then any continuous function $\varphi\colon X \to \mathbb{R}^m$ is uniformly continuous.

Proof

Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 2.5). Moreover $\mathbf{p} \in X$, since X is closed.

The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since

$$\lim_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0.$$

But then the sequences

$$\varphi(\mathbf{u}_{j_1}), \varphi(\mathbf{u}_{j_2}), \varphi(\mathbf{u}_{j_3}), \ldots$$

and

$$\varphi(\mathbf{v}_{j_1}), \varphi(\mathbf{v}_{j_2}), \varphi(\mathbf{v}_{j_3}), \ldots$$

would converge to $\varphi(\mathbf{p})$, since f is continuous (Lemma 2.7), and thus

$$\lim_{k \to +\infty} |\varphi(\mathbf{u}_{j_k}) - \varphi(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$ for all j. We conclude therefore that there must exist some $\delta > 0$ such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎