MA3486 Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2016
Lecture 7 (February 1, 2016)

David R. Wilkins

2.7. Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. We recall that the function φ is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|\varphi(\mathbf{u}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{u}$ of X satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $\varphi\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(\varphi(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $\varphi(\mathbf{p})$ respectively).

Given any function $\varphi\colon X \to Y$, we denote by $\varphi^{-1}(V)$ the *preimage* of a subset V of Y under the map φ, defined by $\varphi^{-1}(V) = \{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$.

Proposition 2.17

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. The function φ is continuous if and only if $\varphi^{-1}(V)$ is open in X for every subset V of Y that is open in Y.

Proof

Suppose that $\varphi\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $\varphi^{-1}(V)$ is open in X. Let $\mathbf{p} \in \varphi^{-1}(V)$. Then $\varphi(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(\varphi(\mathbf{p}), \varepsilon) \subset V$. But φ is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $\varphi(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(V)$. This shows that $\varphi^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $\varphi\colon X \to Y$ is a function with the property that $\varphi^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that φ is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_X(\varphi(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 2.11, hence $\varphi^{-1}(B_Y(\varphi(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(B_Y(\varphi(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$. We conclude that φ is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 2.18

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.

Proof

Let F be a subset of Y that is closed in Y, and let let $V = Y \setminus F$. Then V is open in Y. It follows from Proposition 2.17 that $\varphi^{-1}(V)$ is open in X. But

$$\varphi^{-1}(V) = \varphi^{-1}(Y \setminus F) = X \setminus \varphi^{-1}(F).$$

Indeed let $\mathbf{x} \in X$. Then

$$\begin{aligned} & \mathbf{x} \in \varphi^{-1}(V) \\ \iff & \mathbf{x} \in \varphi^{-1}(Y \setminus F) \\ \iff & \varphi(\mathbf{x}) \in Y \setminus F \\ \iff & \varphi(\mathbf{x}) \notin F \\ \iff & \mathbf{x} \notin \varphi^{-1}(F) \\ \iff & \mathbf{x} \in X \setminus \varphi^{-1}(F). \end{aligned}$$

It follows that the complement $X \setminus \varphi^{-1}(F)$ of $\varphi^{-1}(F)$ in X is open in X, and therefore $\varphi^{-1}(F)$ itself is closed in X, as required. $\blacksquare$

Proposition 2.19

Let $\varphi\colon X \to \mathbb{R}^m$ be a function mapping a subset X of $\mathbb{R}^n$ into $\mathbb{R}^m$. Let $F_1, F_2, \ldots, F_k$ be a finite collection of subsets of X such that F_i is closed in X for $i = 1, 2, \ldots, k$ and

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ is continuous on X if and only if the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Proof

Suppose that $\varphi\colon X \to \mathbb{R}^m$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi\colon X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi\colon X \to \mathbb{R}^m$ is continuous, as required. ∎