MA3486 Fixed Point Theorems and Economic Equilibria
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1.6. Cauchy's Criterion for Convergence

Definition

A sequence $x_1, x_2, x_3, \dots$ of real numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|x_j - x_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.*

Lemma 1.6

Every Cauchy sequence of real numbers is bounded.

Proof

Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence. Then there exists some positive integer N such that $|x_j - x_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|x_j| \leq |x_N| + 1$ whenever $j \geq N$. Therefore $|x_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|x_1|, |x_2|, \ldots, |x_{N-1}|$ and $|x_N| + 1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence*.

Theorem 1.7 (Cauchy's Criterion for Convergence)

An infinite sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof

First we show that convergent sequences are Cauchy sequences. Let $x_1, x_2, x_3, \ldots$ be a convergent sequence of real numbers, and let $l = \lim_{j \to +\infty} x_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - l| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|x_j - l| < \frac{1}{2}\varepsilon$ and $|x_k - l| < \frac{1}{2}\varepsilon$, and hence

$$|x_j - x_k| = |(x_j - l) - (x_k - l)| \leq |x_j - l| + |x_k - l| < \varepsilon.$$

Thus the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $x_1, x_2, x_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 1.6. The sequence $x_1, x_2, x_3, \ldots$ therefore has a convergent subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 1.5). Let $l = \lim_{j \to +\infty} x_{k_j}$. We claim that the sequence $x_1, x_2, x_3, \ldots$ itself converges to l.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - x_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|x_{k_m} - l| < \frac{1}{2}\varepsilon$. Then

$$|x_j - l| \leq |x_j - x_{k_m}| + |x_{k_m} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $x_j \to l$ as $j \to +\infty$, as required. ∎

2. Real Analysis in Euclidean Spaces

2.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 2.1 (Schwarz's Inequality)

The inequality $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ is satisfied by all elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$.

Proof

We note that $|\lambda \mathbf{x} + \mu \mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda \mathbf{x} + \mu \mathbf{y}|^2 = (\lambda \mathbf{x} + \mu \mathbf{y}).(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2.$$

Therefore $\lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4 |\mathbf{x}|^2 - 2|\mathbf{y}|^2 (\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2 |\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right) |\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Corollary 2.2 (Triangle Inequality)

The inequality $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ *is satisfied for all elements* $\mathbf{x}$ *and* $\mathbf{y}$ *of* $\mathbb{R}^n$.

Proof

Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 2.2) that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2. Convergence of Sequences in Euclidean Spaces

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number* ε *satisfying* $\varepsilon > 0$ *there exists some positive integer* N *such that* $|\mathbf{p} - \mathbf{x}_j| < \varepsilon$ *whenever* $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 2.3

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.

Proof

Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be a *Cauchy sequence* if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$.*

Lemma 2.4

A sequence of points in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.

Proof

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$ converging to some point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If $j \geq N$ and $k \geq N$ then

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by the Triangle Inequality. Thus every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Now let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then the ith components of the elements of this sequence constitute a Cauchy sequence of real numbers. This Cauchy sequence must converge to some real number p_i, by Cauchy's Criterion for Convergence (Theorem 1.7). It follows from Lemma 2.3 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. ∎

2.3. The Multidimensional Bolzano-Weierstrass Theorem

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be *bounded* if there exists some constant K such that $|\mathbf{x}_j| \leq K$ for all j.

Example

Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\ (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between -1 and 1. Moreover $x_j = 0$ whenever j is the square of a positive integer, $y_j = 1$ whenever j is even and $z_j = 1$ whenever j is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those x_j for which j is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and -1. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those x_j for which j is the square of an even positive integer.

The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when j is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer p.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those x_j for which j is equal to two raised to the power of an even positive integer.

Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4), \ (x_{16}, y_{16}, z_{16}), \ (x_{64}, y_{64}, z_{64}), \ (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and it therefore follows from Lemma 2.3 that this sequence is a convergent subsequence of the given sequence of points in $\mathbb{R}^3$.

Example

Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$.

Then the first components x_j for which the index j is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components y_j for which the index j is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices j for which x_j is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and y_j is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index j for which x_j and y_j both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded sequence of points in $\mathbb{R}^2$.

Theorem 2.5 (The Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.

Proof

We prove the result by induction on the dimension n of the Euclidean space $\mathbb{R}^n$ that contains the infinite sequence in question. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) that the theorem is true when $n = 1$. Suppose that $n > 1$, and that every bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of elements of $\mathbb{R}^n$, and let $x_{j,i}$ denote the ith component of $\mathbf{x}_j$ for $i = 1, 2, \ldots, n$ and for all positive integers j.

The induction hypothesis requires that all bounded sequences in $\mathbb{R}^{n-1}$ contain convergent subsequences. It follows that there exist real numbers $p_1, p_2, \ldots, p_{n-1}$ and an increasing sequence $m_1, m_2, m_3, \ldots$ of positive integers such that $\lim_{k \to +\infty} x_{m_k,i} = p_i$ for $i = 1, 2, \ldots, n-1$. The nth components $x_{m_1,n}, x_{m_2,n}, x_{m_3,n}, \ldots$ of the members of the subsequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ then constitute a bounded sequence of real numbers. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) that there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers for which the sequence $x_{m_{k_1},n}, x_{m_{k_2},n}, x_{m_{k_3},n}, \ldots$ converges.

Let $s_j = m_{k_j}$ for all positive integers j, and let

$$p_n = \lim_{j \to +\infty} x_{m_{k_j}, n} = \lim_{j \to +\infty} x_{s_j, n}.$$

Then the sequence $x_{s_1, i}, x_{s_2, i}, x_{s_3, i}, \ldots$ converges for values of i between 1 and $n - 1$, because it is a subquence of the convergent sequence

$$x_{m_1, i}, x_{m_2, i}, x_{m_3, i}, \ldots.$$

Moreover

$$x_{s_1, n}, x_{s_2, n}, x_{s_3, n}, \ldots$$

also converges. Thus the ith components of the infinite sequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ converge for $i = 1, 2, \ldots, n$. It then follows from Lemma 2.3 that $\lim\limits_{j \to +\infty} \mathbf{x}_{s_k} = \mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The result follows. ∎