MA3486 Fixed Point Theorems and Economic Equilibria
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1.4.

Definition

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$

$$x_1, x_4, x_9, x_{16}, \ldots$$

1.5. The Bolzano-Weierstrass Theorem

Proposition 1.4

Let $x_1, x_2, x_3, \ldots$ be a bounded infinite sequence of real numbers. Then there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$.

Proof

The infinite sequence $(x_j : j \in \mathbb{N})$ is bounded, and therefore there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Let S denote the set of all real numbers s with the property that

$$\{j \in \mathbb{N} : x_j > s\}$$

is an infinite set. Let $c = \sup S$ (so that c is the least upper bound of the set S).

Let u and v be real numbers satisfying $u < c < v$. Choose v' satisfying $c < v' < v$. Then $v' \notin S$, and therefore

$$\{j \in \mathbb{N} : x_j > v'\}$$

is a finite set. It follows that

$$\{j \in \mathbb{N} : x_j \geq v\}$$

is also a finite set.

Also u is not an upper bound for the set S (because c is the least upper bound, and therefore there exists $u' \in S$ satisfying $u' > u$. It then follows that

$$\{j \in \mathbb{N} : x_j > u'\}$$

is an infinite set, and therefore

$$\{j \in \mathbb{N} : x_j > u\}$$

is an infinite set. But then

$$\{j \in \mathbb{N} : u < x_j < v\}$$

must be an infinite set, since it is obtained by removing from $\{j \in \mathbb{N} : x_j > u\}$ a finite number of values of j for which $x_j \geq v$. The result therefore follows on taking $u = c - \varepsilon$ and $v = c + \varepsilon$. ∎

Theorem 1.5 (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.

First Proof

Let $x_1, x_2, x_3, \ldots$ be an bounded infinite sequence of real numbers. It follows from Proposition 1.4 that there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$. There then exists some positive integer k_1 such that $c - 1 < x_{k_1} < c + 1$.

Now suppose that positive integers $k_1, k_2, \ldots, k_m$ have been determined such that $k_1 < k_2 < \cdots < k_m$ and

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for $j = 1, 2, \ldots, m$. The interval

$$\left\{ x \in \mathbb{R} : c - \frac{1}{m+1} < x < c + \frac{1}{m+1} \right\}$$

must then contain infinitely many members of the original sequence, and therefore there exists some positive integer k_{m+1} for which $k_m < k_{m+1}$ and

$$c - \frac{1}{m+1} < x_{k_{m+1}} < c + \frac{1}{m+1}.$$

Thus we can construct in this fashion a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ of the original sequence with the property that

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for all positive integers j. This subsequence then converges to c. The given sequence therefore has a convergent subsequence, as required. ∎

Second Proof

Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers, and let

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}$$

(i.e., S is the set of all positive integers j with the property that a_j is greater than or equal to all the succeeding members of the sequence).

First let us suppose that the set S is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the manner in which the set S was defined that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.3 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S is finite. Choose a positive integer j_1 which is greater than every positive integer belonging to S. Then j_1 does not belong to S. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 does not belong to S (since j_2 is greater than j_1 and j_1 is greater than every positive integer belonging to S). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.3. This completes the proof of the Bolzano-Weierstrass Theorem. ∎