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1. Ordered Fields and the Real Number System

1.1. Ordered Fields

The real numbers can be characterized as an ordered field with the Least Upper Bound Property. We give below the definition of ordered fields, and then described the Least Upper Bound Property, which requires that every non-empty subset of the set of real numbers that is bounded above has a least upper bound.

An *ordered field* $\mathbb{F}$ consists of a set $\mathbb{F}$ on which are defined binary operations $+$ of addition and $\times$ of multiplication, together with an ordering relation $<$, where these binary operations and ordering relation satisfy the following axioms:—

1. if u and v are elements of $\mathbb{F}$ then their sum $u + v$ is also a element of $\mathbb{F}$;
2. (*the Commutative Law for addition*) $u + v = v + u$ for all elements u and v of $\mathbb{F}$;
3. (*the Associative Law for addition*) $(u + v) + w = u + (v + w)$ for all elements u, v and w of $\mathbb{F}$;
4. there exists an element of $\mathbb{F}$, denoted by 0, with the property that $u + 0 = x = 0 + u$ for all elements u of $\mathbb{F}$;
5. for each element u of $\mathbb{F}$ there exists some element $-u$ of $\mathbb{F}$ with the property that $u + (-u) = 0 = (-u) + u$;

6. if u and v are elements of $\mathbb{F}$ then their product $u \times v$ is also a element of $\mathbb{F}$;
7. (*the Commutative Law for multiplication*) $u \times v = v \times u$ for all elements u and v of $\mathbb{F}$;
8. (*the Associative Law for multiplication*) $(u \times v) \times w = u \times (v \times w)$ for all elements u, v and w of $\mathbb{F}$,
9. there exists an element of $\mathbb{F}$, denoted by 1, with the property that $u \times 1 = u = 1 \times u$ for all elements u of $\mathbb{F}$, and moreover $1 \neq 0$,
10. for each element u of $\mathbb{F}$ satisfying $u \neq 0$ there exists some element u^{-1} of $\mathbb{F}$ with the property that $u \times u^{-1} = 1 = u^{-1} \times u$,

11. (*the Distributive Law*) $u \times (v + w) = (u \times v) + (u \times w)$ for all elements u, v and w of $\mathbb{F}$,
12. (*the Trichotomy Law*) if u and v are elements of $\mathbb{F}$ then one and only one of the three statements $u < v$, $u = v$ and $u < v$ is true,
13. (*transitivity of the ordering*) if u, v and w are elements of $\mathbb{F}$ and if $u < v$ and $v < w$ then $u < w$,
14. if u, v and w are elements of $\mathbb{F}$ and if $u < v$ then $u + w < v + w$,
15. if u and v are elements of $\mathbb{F}$ which satisfy $0 < u$ and $0 < v$ then $0 < u \times v$,

The operations of subtraction and division are defined on an ordered field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $u - v = u + (-v)$ for all elements u and v of $\mathbb{F}$, and moreover $u/v = uv^{-1}$ provided that $v \neq 0$.

Example

The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Let D be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition

Let $\mathbb{F}$ be an ordered field, and let D be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example

The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above that characterize ordered fields are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid in any ordered field whatsoever, and in particular would be valid were the system of real numbers replaced by the system of rational numbers.) We require as an additional axiom the following property.

The Least Upper Bound Property

given any non-empty set D of real numbers that is bounded above, there exists a real number $\sup D$ that is the least upper bound for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed
$\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark

We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techiques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second.

Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convegence criterion. It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.

1.3. Infinite Sequences of Real Numbers

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.*

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim\limits_{j\to+\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l - \varepsilon < x_j < l + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition

We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.1

Every convergent sequence of real numbers is bounded.

Proof

Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number l. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $l - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $l + 1$. ∎

Proposition 1.2

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum, difference and product of these sequences are convergent, and

$$\begin{aligned}
\lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j y_j) &= \left(\lim_{j \to +\infty} x_j \right) \left(\lim_{j \to +\infty} y_j \right).
\end{aligned}$$

If in addition $y_j \neq 0$ for all positive integers j and $\lim_{j \to +\infty} y_j \neq 0$, then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j \to +\infty} \frac{x_j}{y_j} = \frac{\lim_{j \to +\infty} x_j}{\lim_{j \to +\infty} y_j}.$$

Proof

Throughout this proof let $l = \lim_{j \to +\infty} x_j$ and $m = \lim_{j \to +\infty} y_j$.

First we prove that $x_j + y_j \to l + m$ as $j \to +\infty$. Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (l + m)| < \varepsilon$ whenever $j \geq N$.

Now $x_j \to l$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - l| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - l| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - m| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (l + m)| &= |(x_j - l) + (y_j - m)| \leq |x_j - l| + |y_j - m| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to l + m$ as $j \to +\infty$.

Let c be some real number. We show that $cy_j \to cm$ as $j \to +\infty$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|y_j - m| < \varepsilon/|c|$ whenever $j \geq N$. But then $|cy_j - cm| = |c||y_j - m| < \varepsilon$ whenever $j \geq N$. Thus $cy_j \to cm$ as $j \to +\infty$.

If we combine this result, for $c = -1$, with the previous result, we see that $-y_j \to -m$ as $j \to +\infty$, and therefore $x_j - y_j \to l - m$ as $j \to +\infty$.

Next we show that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences, and if $u_j \to 0$ and $v_j \to 0$ as $j \to +\infty$, then $u_j v_j \to 0$ as $j \to +\infty$. Let some strictly positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. We deduce that $u_j v_j \to 0$ as $j \to +\infty$.

We can apply this result with $u_j = x_j - l$ and $v_j = y_j - m$ for all positive integers j. Using the results we have already obtained, we see that

$$\begin{aligned}
0 &= \lim_{j\to+\infty}(u_j v_j) = \lim_{j\to+\infty}(x_j y_j - x_j m - l y_j + lm) \\
&= \lim_{j\to+\infty}(x_j y_j) - m \lim_{j\to+\infty} x_j - l \lim_{j\to+\infty} y_j + lm \\
&= \lim_{j\to+\infty}(x_j y_j) - lm.
\end{aligned}$$

Thus $x_j y_j \to lm$ as $j \to +\infty$.

Next we show that if $w_1, w_2, w_3, \ldots$ is an infinite sequence of non-zero real numbers, and if $w_j \to 1$ as $j \to +\infty$ then $1/w_j \to 1$ as $j \to +\infty$. Let some strictly positive real number ε be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some positive integer N such that $|w_j - 1| < \varepsilon_0$ whenever $j \geq N$. Thus if $j \geq N$ then $|w_j - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < w_j < \frac{3}{2}$. But then

$$\left|\frac{1}{w_j} - 1\right| = \left|\frac{1 - w_j}{w_j}\right| = \frac{|w_j - 1|}{|w_j|} < 2|w_j - 1| < \varepsilon.$$

We deduce that $1/w_j \to 1$ as $j \to +\infty$.

Finally suppose that $\lim_{j \to +\infty} x_j = l$ and $\lim_{j \to +\infty} y_j = m$, where $m \neq 0$. Let $w_j = y_j/m$. Then $w_j \to 1$ as $j \to +\infty$, and hence $1/w_j \to 1$ as $j \to +\infty$. We see therefore that $m/y_j \to 1$, and thus $1/y_j \to 1/m$, as $j \to +\infty$. The result we have already obtained for products of sequences then enables us to deduce that $x_j/y_j \to l/m$ as $j \to +\infty$. ∎

1.4. Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.3

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. $\blacksquare$