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#### **Proposition 5.17**

Let n be a positive integer, let I be a non-empty finite set, let  $\varphi \colon \mathbb{R}^n \to \mathbb{R}$  be a linear functional on  $\mathbb{R}^n$ , and, for each  $i \in I$ , let  $\eta_i \colon \mathbb{R}^n \to \mathbb{R}$  be a linear functional on  $\mathbb{R}^n$ . Suppose that  $\varphi(\mathbf{v}) \ge 0$  for all  $\mathbf{v} \in \mathbb{R}^n$  with the property that  $\eta_i(\mathbf{v}) \ge 0$  for all  $i \in I$ . Then there exist non-negative real numbers  $g_i$  for all  $i \in I$  such that  $\varphi = \sum_{i \in I} g_i \eta_i$ .

## Proof

We may suppose that  $I = \{1, 2, ..., m\}$  for some positive integer m. For each  $i \in I$  there exist real numbers  $A_{i,1}, A_{i,2}, ..., A_{i,n}$  such that

$$\eta_i(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)=\sum_{j=1}^n A_{i,j}\mathbf{v}_j$$

for i = 1, 2, ..., m and for all real numbers  $v_1, v_2, ..., v_n$ . Let A be the  $m \times n$  matrix whose coefficient in the *i*th row and *j*th column is the real number  $A_{i,j}$  for i = 1, 2, ..., m and j = 1, 2, ..., n. Then an *n*-dimensional vector  $\mathbf{v} \in \mathbb{R}^n$  satisfies  $\eta_i(\mathbf{v}) \ge 0$  for all  $i \in I$  if and only if  $A\mathbf{v} \ge \mathbf{0}$ . There exists an *n*-dimensional vector  $\mathbf{c} \in \mathbb{R}^n$  such that  $\varphi(\mathbf{v}) = \mathbf{c}^T \mathbf{v}$  for all  $\mathbf{v} \in \mathbb{R}^n$ . Then  $\mathbf{c}^T \mathbf{v} \ge 0$  for all  $\mathbf{v} \in \mathbb{R}^n$  satisfying  $A\mathbf{v} \ge 0$ . It then follows from Corollary 5.16 that there exists  $\mathbf{y} \in \mathbb{R}^m$  such that  $\mathbf{y}^T A = \mathbf{c}^T$  and  $\mathbf{y} \ge \mathbf{0}$ . Let  $g_i = (\mathbf{y})_i$  for i = 1, 2, ..., m. Then  $g_i \ge 0$  for i = 1, 2, ..., m and  $\sum_{i \in I} g_i \eta_i = \varphi$ , as required.

### Remark

The result of Proposition 5.17 can also be viewed as a consequence of Proposition 5.13 applied to the convex cone in the dual space  $\mathbb{R}^{n*}$  of the real vector space  $\mathbb{R}^{n}$  generated by the linear functionals  $\eta_i$  for  $i \in I$ . Indeed let *C* be the subset of  $\mathbb{R}^{n*}$  defined such that

$$C = \left\{ \sum_{i \in I} g_i \eta_i : g_i \ge 0 \text{ for all } i \in I 
ight\}.$$

It follows from Proposition 5.12 that C is a closed convex cone in the dual space  $\mathbb{R}^{n*}$  of  $\mathbb{R}^n$ . If the linear functional  $\varphi$  did not belong to this cone then it would follow from Proposition 5.13 that there would exist a linear functional  $V \colon \mathbb{R}^{n*} \to \mathbb{R}$  with the property that  $V(\eta_i) \ge 0$  for all  $i \in I$  and  $V(\varphi) < 0$ .

But given any linear functional on the dual space of a given finite-dimensional vector space, there exists some vector belonging to the given vector space such that the linear functional on the dual space evaluates elements of the dual space at that vector (see Corollary 2.7). It follows that there would exist  $\mathbf{v} \in \mathbb{R}^n$  such that  $V(\psi) = \psi(\mathbf{v})$  for all  $\psi \in \mathbb{R}^{n*}$ . But then  $\eta_i(\mathbf{v}) \ge 0$  for all  $i \in I$  and  $\varphi(\mathbf{v}) < 0$ . This contradicts the requirement that  $\varphi(\mathbf{v}) \ge 0$  for all  $\mathbf{v} \in \mathbb{R}^n$  satisfying  $\eta_i(\mathbf{v}) \geq 0$  for all  $i \in I$ . To avoid this contradiction it must be the case that  $\varphi \in C$ , and therefore there must exist non-negative real numbers  $g_i$  for all  $i \in I$  such that  $\varphi = \sum_{i \in I} g_i \eta_i.$ 

# Corollary 5.18

Let n be a positive integer, let I be a non-empty finite set, let  $\varphi : \mathbb{R}^n \to \mathbb{R}$  be a linear functional on  $\mathbb{R}^n$ , and, for each  $i \in I$ , let  $\eta_i : \mathbb{R}^n \to \mathbb{R}$  be a linear functional on  $\mathbb{R}^n$ . Suppose that there exists a subset  $I_0$  of I such that  $\varphi(\mathbf{v}) \ge 0$  for all  $\mathbf{v} \in \mathbb{R}^n$  with the property that  $\eta_i(\mathbf{v}) \ge 0$  for all  $i \in I_0$ . Then there exist non-negative real numbers  $g_i$  for all  $i \in I$  such that  $\varphi = \sum_{i \in I} g_i \eta_i$  and  $g_i = 0$  when  $i \notin I_0$ .

### Proof

It follows directly from Proposition 5.17 that there exist non-negative real numbers  $g_i$  for all  $i \in I_0$  such that  $\varphi = \sum_{i \in I_0} g_i \eta_i$ . Let  $g_i = 0$  for all  $i \in I \setminus I_0$ . Then  $\varphi = \sum_{i \in I_0} g_i \eta_i$ , as required.

## Definition

A subset X is said to be a *convex polytope* if there exist linear functionals  $\eta_1, \eta_2, \ldots, \eta_m$  on  $\mathbb{R}^n$  and real numbers  $s_1, s_2, \ldots, s_m$  such that

$$X = \{\mathbf{x} \in \mathbb{R}^n : \eta_i(\mathbf{x}) \ge s_i \text{ for } i = 1, 2, \dots, m\}.$$

Let  $(\eta_i : i \in I)$  be a finite collection of linear functionals on  $\mathbb{R}^n$ indexed by a finite set I, let  $s_i$  be a real number for all  $i \in I$ , and let

$$X = \bigcap_{i \in I} \{ \mathbf{x} \in \mathbb{R} : \eta_i(\mathbf{x}) \ge s_i \}.$$

Then X is a convex polytope in  $\mathbb{R}^n$ . A point **x** of  $\mathbb{R}^n$  belongs to the convex polytope X if and only if  $\eta_i(\mathbf{x}) \ge s_i$  for all  $i \in I$ .

## **Proposition 5.19**

Let n be a positive integer, let I be a non-empty finite set, and, for each  $i \in I$ , let  $\eta_i \colon \mathbb{R}^n \to \mathbb{R}$  be non-zero linear functional and let  $s_i$ be a real number. Let X be the convex polytope defined such that

$$X = \bigcap_{i \in I} \{ \mathbf{x} \in \mathbb{R} : \eta_i(\mathbf{x}) \ge s_i \}.$$

(Thus a point  $\mathbf{x}$  of  $\mathbb{R}^n$  belongs to the convex polytope X if and only if  $\eta_i(\mathbf{x}) \ge s_i$  for all  $i \in I$ .) Let  $\varphi \colon \mathbb{R}^n \to \mathbb{R}$  be a non-zero linear functional on  $\mathbb{R}^n$ , and let  $\mathbf{x}^* \in X$ . Then  $\varphi(\mathbf{x}^*) \le \varphi(\mathbf{x})$  for all  $\mathbf{x} \in X$  if and only if there exist non-negative real numbers  $g_i$  for all  $i \in I$  such that  $\varphi = \sum_{i \in I} g_i \eta_i$  and  $g_i = 0$  whenever  $\eta_i(\mathbf{x}^*) > s_i$ .

#### Proof

Let  $K = \{i \in I : \eta_i(\mathbf{x}^*) > s_i\}$ . Suppose that there do not exist non-negative real numbers  $g_i$  for all  $i \in I$  such that  $\varphi = \sum_{i \in I} g_i \eta_i$ and  $g_i = 0$  when  $i \in K$ . Corollary 5.18 then ensures that there must exist some  $\mathbf{v} \in \mathbb{R}^n$  such that  $\eta_i(\mathbf{v}) \ge 0$  for all  $i \in I \setminus K$  and  $\varphi(\mathbf{v}) < 0$ . Then

$$\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) = \eta_i(\mathbf{x}^*) + \lambda \eta_i(\mathbf{v}) \ge s_i$$

for all  $i \in I \setminus K$  and for all  $\lambda \ge 0$ . If  $i \in K$  then  $\eta_i(\mathbf{x}^*) > s_i$ . The set K is finite. It follows that there must exist some real number  $\lambda_0$  satisfying  $\lambda_0 > 0$  such that  $\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) \ge s_i$  for all  $i \in K$  and for all real numbers  $\lambda$  satisfying  $0 \le \lambda \le \lambda_0$ .

Combining the results in the cases when  $i \in I \setminus K$  and when  $i \in K$ , we find that  $\eta_i(\mathbf{x}^* + \lambda \mathbf{v}) \geq s_i$  for all  $i \in I$  and  $\lambda \in [0, \lambda_0]$ , and therefore  $\mathbf{x}^* + \lambda \mathbf{v} \in X$  for all real numbers  $\lambda$  satisfying  $0 \leq \lambda \leq \lambda_0$ . But

$$\varphi(\mathbf{x}^* + \lambda \mathbf{v}) = \varphi(\mathbf{x}^*) + \lambda \varphi(\mathbf{v}) < \varphi(\mathbf{x}^*)$$

whenever  $\lambda > 0$ . It follows that the linear functional  $\varphi$  cannot attain a minimum value in X at any point  $\mathbf{x}^*$  for which either K = I or for which K is a proper subset of I but there exist non-negative real numbers  $g_i$  for all  $i \in I \setminus K$  such that  $\varphi = \sum_{i \in I \setminus K} g_i \eta_i$ . The result follows.

# 5.8. Strong Duality

# Example

Consider again the following linear programming problem in general primal form:—

find values of  $x_1$ ,  $x_2$ ,  $x_3$  and  $x_4$  so as to minimize the objective function

 $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ 

subject to the following constraints:---

•  $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1;$ •  $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 = b_2;$ •  $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \ge b_3;$ •  $x_1 \ge 0$  and  $x_3 \ge 0.$  Now the constraint

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$$

can be expressed as a pair of inequality constraints as follows:

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 & \geq & b_1 \\ -a_{1,1}x_1 - a_{1,2}x_2 - a_{1,3}x_3 - a_{1,4}x_4 & \geq & -b_1. \end{array}$$

Similarly the equality constraint involving  $b_2$  can be expressed as a pair or inequality constraints.

## 5. Duality and Convexity (continued)

Therefore the problem can be reformulated as follows:----

find values of  $x_1$ ,  $x_2$ ,  $x_3$  and  $x_4$  so as to minimize the objective function

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to the following constraints:-

• 
$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 \ge b_1;$$
  
•  $-a_{1,1}x_1 - a_{1,2}x_2 - a_{1,3}x_3 - a_{1,4}x_4 \ge -b_1;$   
•  $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 \ge b_2;$   
•  $-a_{2,1}x_1 - a_{2,2}x_2 - a_{2,3}x_3 - a_{2,4}x_4 \ge -b_2;$   
•  $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \ge b_3;$   
•  $x_1 \ge 0;$   
•  $x_3 \ge 0.$ 

Let

$$\varphi(x_1, x_2, x_3, x_4) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4,$$

and let

$$\begin{aligned} \eta_1^+(x_1, x_2, x_3, x_4) &= a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4, \\ \eta_1^-(x_1, x_2, x_3, x_4) &= -\eta_1(x_1, x_2, x_3, x_4), \\ \eta_2^+(x_1, x_2, x_3, x_4) &= a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4, \\ \eta_2^-(x_1, x_2, x_3, x_4) &= -\eta_3(x_1, x_2, x_3, x_4), \\ \eta_3(x_1, x_2, x_3, x_4) &= a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4, \\ \zeta_1(x_1, x_2, x_3, x_4) &= x_1, \\ \zeta_3(x_1, x_2, x_3, x_4) &= x_3, \end{aligned}$$

Then  $(x_1, x_2, x_3, x_4)$  is a feasible solution to the primal problem if and only if this element of  $\mathbb{R}^4$  belongs to the convex polytope X, where X is the subset of  $\mathbb{R}^4$  consisting of all points **x** of  $\mathbb{R}^4$  that satisfy the following constraints:—

- $\eta_1^+({\sf x}) \ge b_1;$
- $\eta_1^-(\mathbf{x}) \geq -b_1;$
- $\eta_2^+(\mathbf{x}) \ge b_2;$
- $\eta_2^-(\mathbf{x}) \ge -b_2;$
- $\eta_3(\mathbf{x}) \geq b_3;$
- $\zeta_1(\mathbf{x}) \geq 0;$
- $\zeta_3(\mathbf{x}) \ge 0.$

An inequality constraint is said to be *binding* for a particular feasible solution **x** if equality holds in that constraint at the feasible solution. Thus the constraints on the values of  $\eta_1^+$ ,  $\eta_1^-$ ,  $\eta_2^+$  and  $\eta_2^-$  are always binding at points of the convex polytope X, but the constraints determined by  $\eta_3$ ,  $\zeta_1$  and  $\zeta_3$  need not be binding.

Suppose that the linear functional  $\varphi$  attains its minimum value at a point  $\mathbf{x}^*$  of X, where  $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ . It then follows from Proposition 5.19 that there exist non-negative real numbers  $p_1^+$ ,  $p_1^-$ ,  $p_2^+$ ,  $p_2^-$ ,  $p_3$ ,  $q_1$  and  $q_3$  such that

$$p_1^+\eta_1^+ + p_1^-\eta_1^- + p_2^+\eta_2^+ + p_2^-\eta_2^- + p_3\eta_3 + q_1\zeta_1 + q_3\zeta_3 = \varphi.$$

Moreover  $p_3 = 0$  if  $\eta_3(\mathbf{x}^*) > b_3$ ,  $q_1 = 0$  if  $\zeta_1(\mathbf{x}^*) > 0$ , and  $q_3 = 0$  if  $\zeta_3(\mathbf{x}^*) > 0$ .

Now 
$$\eta_1^- = -\eta_1^+$$
 and  $\eta_2^- = -\eta_2^+$ . It follows that  
 $p_1\eta_1^+ + p_2\eta_2^+ + p_3\eta_3 + q_1\zeta_1 + q_3\zeta_3 = \varphi$ ,  
where  $p_1 = p_1^+ - p_1^-$  and  $p_2 = p_2^+ - p_2^-$ . Moreover  $p_3 = 0$  if  
 $\sum_{i=1}^4 a_{3,i}x_i^* > b_3$ ,  $q_1 = 0$  if  $x_1^* > 0$ , and  $q_3 = 0$  if  $x_3^* > 0$ .

### It follows that

$$\begin{array}{rcl} p_1 a_{1,1} + p_2 a_{2,1} + p_3 a_{3,1} &\leq & c_1, \\ p_1 a_{1,2} + p_2 a_{2,2} + p_3 a_{3,2} &= & c_2, \\ p_1 a_{1,3} + p_2 a_{2,3} + p_3 a_{3,3} &\leq & c_3, \\ p_1 a_{1,4} + p_2 a_{2,4} + p_3 a_{3,4} &= & c_4, \\ p_3 &\geq & 0. \end{array}$$

Moreover  $p_3 = 0$  if  $\sum_{i=1}^{4} a_{3,j}x_j^* > b_3$ ,  $\sum_{i=1}^{3} p_i a_{i,1} = c_1$  if  $x_1^* > 0$ , and  $\sum_{i=1}^{3} p_i a_{i,3} = c_3$  if  $x_3^* > 0$ . It follows that  $(p_1, p_2, p_3)$  is a feasible solution of the dual problem to the feasible primal problem.

Moreover the complementary slackness conditions determined by the primal problem are satisfied. It therefore follows from the Weak Duality Theorem (Theorem 5.5) that  $(p_1, p_2, p_3)$  is an optimal solution to the dual problem.

# Theorem 5.20

(Strong Duality for Linear Programming Problems with Optimal Solutions)

Let  $\mathbf{x}^* \in \mathbb{R}^n$  be an optimal solution to a linear programming problem

 $\operatorname{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ 

expressed in general primal form with constraint matrix A with m rows and n columns, target vector **b**, cost vector **c**, inequality constraint specifier  $I^+$  and variable sign specifier  $J^+$ . Then there exists an optimal solution **p**<sup>\*</sup> to the corresponding dual programming problem

 $\mathrm{Dual}(A,\mathbf{b},\mathbf{c},I^+,J^+),$ 

and moreover  $\mathbf{p}^{*T}\mathbf{b} = \mathbf{c}^T \mathbf{x}^*$ .

### Proof

Let  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ , and let  $A_{i,j} = (A)_{i,j}$ ,  $b_i = (\mathbf{b})_i$  and  $c_j = (\mathbf{c})_j$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ . Then optimal solution  $\mathbf{x}^*$  minimizes  $\mathbf{c}^T \mathbf{x}^*$  subject to the following constraints:—

• 
$$A\mathbf{x}^* \geq \mathbf{b};$$

• 
$$(A\mathbf{x}^*)_i = b_i$$
 unless  $i \in I^+$ ;

• 
$$x_j^* \ge 0$$
 for all  $j \in J^+$ .

Let **p** be a feasible solution to the dual linear programming problem, where  $\mathbf{p} = (p_1, p_2, \dots, p_m)$ . Then **p** must satisfy the following constraints:—

- $\mathbf{p}^T A \leq \mathbf{c}^T$ ;
- $p_i \ge 0$  for all  $i \in I^+$ ;

• 
$$(\mathbf{p}^T A)_j = c_j$$
 unless  $j \in J^+$ .

Now the constraints of the primal problem can be expressed in inequality form as follows:----

• 
$$(A\mathbf{x}^*)_i \ge b_i$$
 for all  $i \in I^+$ ;  
•  $(A\mathbf{x}^*)_i \ge b_i$  for all  $i \in I \setminus I^+$ ;  $(-A\mathbf{x}^*)_i \ge -b_i$  for all  $i \in I \setminus I^+$ ;  
•  $x_j^* \ge 0$  for all  $j \in J^+$ .  
Let

$$\varphi(x_1, x_2, \ldots, x_n) = \sum_{j=1}^n c_j x_j,$$

$$\eta_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^n (A)_{i,j} x_j \quad (i = 1, 2, \dots, m)$$
  
$$\zeta_j(x_1, x_2, \dots, x_n) = x_j \quad (j = 1, 2, \dots, n)$$

It follows from Proposition 5.19 that if there exists an optimal solution to the primal problem then there exist non-negative quantities  $p_i$  for all  $i \in I^+$ ,  $p_i^+$  and  $p_-$  for all  $i \in I \setminus I^+$  and  $q_j$  for all  $j \in J^+$  such that

$$\varphi = \sum_{i \in I^+} p_i \eta_i + \sum_{i \in I \setminus I^+} (p_i^+ - p_i^-) \eta_i + \sum_{j \in J^+} q_j \zeta_j.$$

Moreover  $p_i = 0$  whenever  $i \in I^+$  and  $\eta_i(x_1^*, x_2^*, \dots, x_n^*)_i > b_i$  and  $q_j = 0$  whenever  $x_j^* > 0$ .

Let  $\mathbf{p}^* \in \mathbb{R}^m$  be defined such that  $(\mathbf{p}^*)_i = p_i$  for all  $i \in I^+$  and  $(\mathbf{p}^*)_i = p_i^+ - p_i^-$  for all  $i \in I \setminus I^+$ . Then  $(\mathbf{p}^{*T}A)_j \leq c_j$  for j = 1, 2, ..., n,  $(\mathbf{p}^*)_i \geq 0$  for all  $i \in I^+$ , and  $(\mathbf{p}^{*T}A)_j = c_j$  unless  $j \in J^+$ . Moreover  $(\mathbf{p}^*)_i = 0$  whenever  $(A\mathbf{x}^*)_i > b_i$  and  $q_i = 0$  whenever  $x_j > 0$ . It follows that  $\mathbf{p}^*$  is a feasible solution of the dual problem. Moreover the relevant complementary slackness conditions are satisfied by  $\mathbf{x}^*$  and  $\mathbf{p}^*$ . It is then a consequence of the Weak Duality Theorem that  $\mathbf{c}^T \mathbf{x}^* = \mathbf{p}^{*T} \mathbf{b}$ , and that therefore  $\mathbf{p}^*$  is an optimal solution of the dual problem (see Corollary 5.6). The result follows.