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5.3. Complementary Slackness and the Weak Duality Theorem

Theorem 5.2

(Weak Duality Theorem for Linear Programming Problems in Dantzig Standard Form)

Let m and n be integers, let A be an $m \times n$ matrix with real coefficients, let $\mathbf{b} \in \mathbb{R}^m$ and let $\mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and let $\mathbf{p} \in \mathbb{R}^m$ satisfy the constraint $\mathbf{p}^T A \leq \mathbf{c}$. Then $\mathbf{p}^T \mathbf{b} \leq \mathbf{c}^T \mathbf{x}$. Moreover $\mathbf{p}^T \mathbf{b} = \mathbf{c}^T \mathbf{x}$ if and only if the following complementary slackness condition is satisfied:

- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *for all integers j between 1 and n for which* $(\mathbf{x})_j > 0$.

Proof

Let $x_j = (\mathbf{x})_j$ and $c_j = (\mathbf{c})_j$ for $j = 1, 2, \ldots, n$. The constraints satisfied by the vectors $\mathbf{x}$ and $\mathbf{p}$ ensure that

$$\begin{aligned} \mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}) \\ &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x}, \end{aligned}$$

because $A\mathbf{x} - \mathbf{b} = \mathbf{0}$. But also $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^T - \mathbf{p}^T A \geq \mathbf{0}$, and therefore $(\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} \geq 0$. Moreover

$$(\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} = \sum_{j=1}^{n} (c_j - (\mathbf{p}^T A)_j) x_j,$$

where $c_j - (\mathbf{p}^T A)_j \geq 0$ and $x_j \geq 0$ for $j = 1, 2, \ldots, n$. It follows that $(\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} = 0$ if and only if $c_j - (\mathbf{p}^T A)_j = 0$ for all integers j between 1 and n for which $x_j > 0$. The result follows. ∎

Corollary 5.3

Let a linear programming problem in Dantzig standard form be specified by an $m \times n$ constraint matrix A, and m-dimensional target vector $\mathbf{b}$ and an n-dimensional cost vector $\mathbf{c}$. Let $\mathbf{x}^*$ be a feasible solution of this primal problem, and let $\mathbf{p}^*$ be a solution of the dual problem. Then ${\mathbf{p}^*}^T A \leq \mathbf{c}^T$. Suppose that the complementary slackness conditions for this primal-dual pair are satisfied, so that $({\mathbf{p}^*}^T A)_j = (\mathbf{c})_j$ for all integers j between 1 and n for which $(\mathbf{x}^*)_j > 0$. Then $\mathbf{x}^*$ is an optimal solution of the primal problem, and $\mathbf{p}^*$ is an optimal solution of the dual problem.

Proof

Because the complementary slackness conditions for this primal-dual pair are satisfied, it follows from the Weak Duality Theorem that $\mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$ (see Theorem 5.2). But it then also follows from the Weak Duality Theorem that

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^{*T}\mathbf{b} = \mathbf{c}^T\mathbf{x}^*$$

for all feasible solutions $\mathbf{x}$ of the primal problem. It follows that $\mathbf{x}^*$ is an optimal solution of the primal problem. Similarly

$$\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$$

for all feasible solutions $\mathbf{p}$ of the dual problem. It follows that $\mathbf{p}^*$ is an optimal solution of the dual problem, as required. ∎

Another special case of duality in linear programming is exemplified by a primal-dual pair of problems in *Von Neumann Symmetric Form*. In this case the primal and dual problems are specified in terms of an $m \times n$ constraint matrix A, an m-dimensional target vector $\mathbf{b}$ and an n-dimensional cost vector $\mathbf{c}$. The objective of the problem is minimize $\mathbf{c}^T\mathbf{x}$ amongst n-dimensional vectors $\mathbf{x}$ that satisfy the constraints $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. The dual problem is to maximize $\mathbf{p}^T\mathbf{b}$ amongst m-dimensional vectors $\mathbf{p}$ that satisfy the constraints $\mathbf{p}^T A \leq \mathbf{c}^T$ and $\mathbf{p} \geq \mathbf{0}$.

Theorem 5.4

(Weak Duality Theorem for Linear Programming Problems in Von Neumann Symmetric Form)

Let m and n be integers, let A be an $m \times n$ matrix with real coefficients, let $\mathbf{b} \in \mathbb{R}^m$ and let $\mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy the constraints $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and let $\mathbf{p} \in \mathbb{R}^m$ satisfy the constraints $\mathbf{p}^T A \leq \mathbf{c}$ and $\mathbf{p}^T \geq \mathbf{0}$. Then $\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$. Moreover $\mathbf{p}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$ if and only if the following complementary slackness conditions are satisfied:

- $(A\mathbf{x})_i = (\mathbf{b})_i$ *for all integers i between 1 and m for which $(\mathbf{p})_i > 0$;*
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *for all integers j between 1 and n for which $(\mathbf{x})_j > 0$;*

Proof

The constraints satisfied by the vectors $\mathbf{x}$ and $\mathbf{p}$ ensure that

$$\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}).$$

But $\mathbf{x} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{0}$, $A\mathbf{x} - \mathbf{b} \geq \mathbf{0}$ and $\mathbf{c}^T - \mathbf{p}^T A \geq \mathbf{0}$. It follows that $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} \geq 0$. and therefore $\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{b}$. Moreover $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = 0$ if and only if $(\mathbf{c}^T - \mathbf{p}^T A)_j(\mathbf{x})_j = 0$ for $j = 1, 2, \ldots, n$ and $(\mathbf{p})_i(A\mathbf{x} - \mathbf{b})_i = 0$, and therefore $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$ if and only if the complementary slackness conditions are satisfied. $\blacksquare$

Theorem 5.5

(Weak Duality Theorem for Linear Programming Problems in General Primal Form)

Let $\mathbf{x} \in \mathbb{R}^n$ *be a feasible solution to a linear programming problem* $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ *expressed in general primal form with constraint matrix* A *with* m *rows and* n *columns, target vector* $\mathbf{b}$*, cost vector* $\mathbf{c}$*, inequality constraint specifier* I^+ *and variable sign specifier* J^+*, and let* $\mathbf{p} \in \mathbb{R}^m$ *be a feasible solution to the corresponding dual programming problem* $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$*. Then* $\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$*. Moreover* $\mathbf{p}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$ *if and only if the following complementary slackness conditions are satisfied:—*

- $(A\mathbf{x})_i = \mathbf{b}_i$ *whenever* $(\mathbf{p})_i \neq 0$*;*
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *whenever* $(\mathbf{x})_j \neq 0$*.*

Proof

The feasible solution $\mathbf{x}$ to the primal problem satisfies the following constraints:—

- $A\mathbf{x} \geq \mathbf{b}$;
- $(A\mathbf{x})_i = (\mathbf{b})_i$ unless $i \in I^+$;
- $(\mathbf{x})_j \geq 0$ for all $j \in J^+$.

The feasible solution $\mathbf{p}$ to the dual problem satisfies the following constraints:—

- $\mathbf{p}^T A \leq \mathbf{c}^T$;
- $(\mathbf{p})_i \geq 0$ for all $i \in I^+$;
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ unless $j \in J^+$.

Now

$$\begin{aligned}
\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}) \\
&= \sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j + \sum_{i=1}^{m} (\mathbf{p})_i (A\mathbf{x} - \mathbf{b})_i.
\end{aligned}$$

Let j be an integer between 1 and n. If $j \in J^+$ then $(\mathbf{x})_j \geq 0$ and $(\mathbf{c}^T - \mathbf{p}^T A)_j \geq 0$, and therefore

$$(\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j \geq 0.$$

If $j \notin J^+$ then $(\mathbf{p}^T A)_j = (\mathbf{c})_j$, and therefore

$$(\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j = 0,$$

irrespective of whether $(\mathbf{x})_j$ is positive, negative or zero.

It follows that

$$\sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j \geq 0.$$

Moreover

$$\sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j = 0$$

if and only if $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ for all indices j for which $(\mathbf{x})_j \neq 0$.

Next let i be an index between 1 and m. If $i \in I^+$ then $(\mathbf{p})_i \geq 0$ and $(A\mathbf{x} - \mathbf{b})_i \geq 0$, and therefore $(\mathbf{p})_i(A\mathbf{x} - \mathbf{b})_i \geq 0$. If $i \notin I^+$ then $(A\mathbf{x})_i = (\mathbf{b})_i$, and therefore $(\mathbf{p})_i(A\mathbf{x} - \mathbf{b})_i = 0$, irrespective of whether $(\mathbf{p})_i$ is positive, negative or zero. It follows that

$$\sum_{i=1}^{m} (\mathbf{p})_i(A\mathbf{x} - \mathbf{p})_i \geq 0.$$

Moreover

$$\sum_{i=1}^{m} (\mathbf{p})_i(A\mathbf{x} - \mathbf{p})_i = 0.$$

if and only if $(A\mathbf{x})_i = (\mathbf{b})_i$ for all indices i for which $(\mathbf{p})_i \neq 0$. The result follows. ∎

Corollary 5.6

Let $\mathbf{x}^* \in \mathbb{R}^n$ be a feasible solution to a linear programming problem $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ expressed in general primal form with constraint matrix A with m rows and n columns, target vector $\mathbf{b}$, cost vector $\mathbf{c}$, inequality constraint specifier I^+ and variable sign specifier J^+, and let $\mathbf{p}^* \in \mathbb{R}^m$ be a feasible solution to the corresponding dual programming problem $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$. Suppose that the complementary slackness conditions are satisfied for this pair of problems, so that $(A\mathbf{x})_i = \mathbf{b}_i$ whenever $(\mathbf{p})_i \neq 0$, and $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ whenever $(\mathbf{x})_j \neq 0$. Then $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{p}^*$ is an optimal solution for the dual problem.

Proof

Because the complementary slackness conditions for this primal-dual pair are satisfied, it follows from the Weak Duality Theorem that $\mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$ (see Theorem 5.5). But it then also follows from the Weak Duality Theorem that

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^{*T}\mathbf{b} = \mathbf{c}^T\mathbf{x}^*$$

for all feasible solutions $\mathbf{x}$ of the primal problem. It follows that $\mathbf{x}^*$ is an optimal solution of the primal problem. Similarly

$$\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$$

for all feasible solutions $\mathbf{p}$ of the dual problem. It follows that $\mathbf{p}^*$ is an optimal solution of the dual problem, as required. ∎

Example

Consider the following linear programming problem in general primal form:—

find values of x_1, x_2, x_3 and x_4 so as to minimize the objective function

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to the following constraints:—

- $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$;
- $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 = b_2$;
- $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \geq b_3$;
- $x_1 \geq 0$ *and* $x_3 \geq 0$.

Here $a_{i,j}$, b_i and c_j are constants for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

The dual problem is the following:—

find values of p_1, p_2 and p_3 so as to maximize the objective function

$$p_1 b_1 + p_2 b_2 + p_3 b_3$$

subject to the following constraints:—

- $p_1 a_{1,1} + p_2 a_{2,1} + p_3 a_{3,1} \leq c_1$;
- $p_1 a_{1,2} + p_2 a_{2,2} + p_3 a_{3,2} = c_2$;
- $p_1 a_{1,3} + p_2 a_{2,3} + p_3 a_{3,3} \leq c_3$;
- $p_1 a_{1,4} + p_2 a_{2,4} + p_3 a_{3,4} = c_4$;
- $p_3 \geq 0$.

We refer to the first and second problems as the *primal problem* and the *dual problem* respectively. Let (x_1, x_2, x_3, x_4) be a feasible solution of the primal problem, and let (p_1, p_2, p_3) be a feasible solution of the dual problem. Then

$$\begin{aligned}\sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i \;&=\; \sum_{j=1}^{4} \left(c_j - \sum_{i=1}^{3} p_i a_{i,j} \right) x_j \\ &\qquad + \sum_{i=1}^{3} p_i \left(\sum_{j=1}^{4} a_{i,j} x_j - b_i \right).\end{aligned}$$

Now the quantity $c_j - \sum_{i=1}^{3} p_i a_{i,j} = 0$ for $j = 2$ and $j = 4$, and $\sum_{j=1}^{4} a_{i,j} x_j - b_i = 0$ for $i = 1$ and $i = 2$. It follows that

$$\begin{aligned}\sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i &= \left(c_1 - \sum_{i=1}^{3} p_i a_{i,1} \right) x_1 \\ &\quad + \left(c_3 - \sum_{i=1}^{3} p_i a_{i,3} \right) x_3 \\ &\quad + p_3 \left(\sum_{j=1}^{4} a_{3,j} x_j - b_3 \right).\end{aligned}$$

Now $x_1 \geq 0$, $x_3 \geq 0$ and $p_3 \geq 0$. Also

$$c_1 - \sum_{i=1}^{3} p_i a_{i,1} \geq 0, \quad c_3 - \sum_{i=1}^{3} p_i a_{i,3} \geq 0$$

and

$$\sum_{j=1}^{4} a_{3,j} x_j - b_3 \geq 0.$$

It follows that

$$\sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i \geq 0.$$

and thus

$$\sum_{j=1}^{4} c_j x_j \geq \sum_{i=1}^{3} p_i b_i.$$

Now suppose that

$$\sum_{j=1}^{4} c_j x_j = \sum_{i=1}^{3} p_i b_i.$$

Then

$$\begin{aligned}
\left(c_1 - \sum_{i=1}^{3} p_i a_{i,1}\right) x_1 &= 0,\\
\left(c_3 - \sum_{i=1}^{3} p_i a_{i,3}\right) x_3 &= 0,\\
p_3 \left(\sum_{j=1}^{4} a_{3,j} x_j - b_3\right) &= 0,
\end{aligned}$$

because a sum of three non-negative quantities is equal to zero if and only if each of those quantities is equal to zero.

It follows that

$$\sum_{j=1}^{4} c_j x_j = \sum_{i=1}^{3} p_i b_i$$

if and only if the following three complementary slackness conditions are satisfied:—

- $\displaystyle\sum_{i=1}^{3} p_i a_{i,1} = c_1$ if $x_1 > 0$;
- $\displaystyle\sum_{i=1}^{3} p_i a_{i,3} = c_3$ if $x_3 > 0$;
- $\sum_{j=1}^{4} a_{3,j} x_j = b_3$ if $p_3 > 0$.

5.4. Open and Closed Sets in Euclidean Spaces

Let m be a positive integer. The *Euclidean norm* $|\mathbf{x}|$ of an element $\mathbf{x}$ of $\mathbb{R}^m$ is defined such that

$$|\mathbf{x}|^2 = \sum_{i=1}^{m} (\mathbf{x})_i^2.$$

The *Euclidean distance function* d on $\mathbb{R}^m$ is defined such that

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. The Euclidean distance function satisfies the Triangle Inequality, together with all the other basic properties required of a distance function on a metric space, and therefore $\mathbb{R}^m$ with the Euclidean distance function is a metric space.

A subset U of $\mathbb{R}^m$ is said to be *open* in $\mathbb{R}^m$ if, given any point $\mathbf{b}$ of U, there exists some real number ε satisfying $\varepsilon > 0$ such that

$$\{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| < \varepsilon\} \subset U.$$

A subset of $\mathbb{R}^m$ is *closed* in $\mathbb{R}^m$ if and only if its complement is open in $\mathbb{R}^m$.

Every union of open sets in $\mathbb{R}^m$ is open in $\mathbb{R}^m$, and every finite intersection of open sets in $\mathbb{R}^m$ is open in $\mathbb{R}^m$.

Every intersection of closed sets in $\mathbb{R}^m$ is closed in $\mathbb{R}^m$, and every finite union of closed sets in $\mathbb{R}^m$ is closed in $\mathbb{R}^m$.

Lemma 5.7

Let m be a positive integer, let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ be a basis of $\mathbb{R}^m$, and let

$$F = \left\{ \sum_{i=1}^{m} s_i \mathbf{u}^{(i)} : s_i \geq 0 \text{ for } i = 1, 2, \ldots, m \right\}.$$

Then F is a closed set in $\mathbb{R}^m$.

Proof

Let $T\colon \mathbb{R}^m \to \mathbb{R}^m$ be defined such that

$$T(s_1, s_2, \ldots, s_m) = \sum_{i=1}^{m} s_i \mathbf{u}^{(i)}$$

for all real numbers $s_1, s_2, \ldots, s_m$. Then T is an invertible linear operator on $\mathbb{R}^m$, and $F = T(G)$, where

$$G = \{\mathbf{x} \in \mathbb{R}^m : (\mathbf{x})_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}.$$

Moreover the subset G of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$.

Now it is a standard result of real analysis that every linear operator on a finite-dimensional vector space is continuous. Therefore $T^{-1}\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous. Moreover $T(G)$ is the preimage of the closed set G under the continuous map T^{-1}, and the preimage of any closed set under a continuous map is itself closed. It follows that $T(G)$ is closed in $\mathbb{R}^m$. Thus F is closed in $\mathbb{R}^m$, as required. $\blacksquare$

Lemma 5.8

Let m be a positive integer, let F be a non-empty closed set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.

Proof

Let R be a positive real number chosen large enough to ensure that the set F_0 is non-empty, where

$$F_0 = F \cap \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| \leq R\}.$$

Then F_0 is a closed bounded subset of $\mathbb{R}^m$. Let $f \colon F_0 \to \mathbb{R}$ be defined such that $f(\mathbf{x}) = |\mathbf{x} - \mathbf{b}|$ for all $\mathbf{x} \in F$. Then $f \colon F_0 \to \mathbb{R}$ is a continuous function on F_0.

Now it is a standard result of real analysis that any continuous real-valued function on a closed bounded subset of a finite-dimensional Euclidean space attains a minimum value at some point of that set. It follows that there exists an element $\mathbf{g}$ of F_0 such that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F_0$. If $\mathbf{x} \in F \setminus F_0$ then

$$|\mathbf{x} - \mathbf{b}| \geq R \geq |\mathbf{g} - \mathbf{b}|.$$

It follows that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$, as required. ∎