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David R. Wilkins

4.8. The Simplex Tableau Algorithm

In describing the Simplex Tableau Algorithm, we adopt notation previously introduced. Thus we are concerned with the solution of a linear programming problem in Dantzig standard form, specified by positive integers m and n, an $m \times n$ constraint matrix A of rank m, a target vector $\mathbf{b} \in \mathbb{R}^m$ and a cost vector $\mathbf{c} \in \mathbb{R}^n$. The optimization problem requires us to find a vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\mathbf{c}^T\mathbf{x}$ amongst all vectors $\mathbf{x} \in \mathbb{R}^n$ that satisfy the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

We denote by $A_{i,j}$ the coefficient in the ith row and jth column of the matrix A, we denote the ith component of the target vector $\mathbf{b}$ by b_i and we denote the jth component of the cost vector $\mathbf{c}$ by c_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

As usual, we define vectors $\mathbf{a}^{(j)} \in \mathbb{R}^m$ for $j = 1, 2, \ldots, n$ such that $(\mathbf{a}^{(j)})_i = A_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n determine a basis B, where

$$B = \{j_1, j_2, \ldots, j_m\},$$

if and only if the corresponding vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ constitute a basis of $\mathbb{R}^m$. Given such a basis B we let M_B denote the invertible $m \times m$ matrix defined such that $(M_B)_{i,k} = A_{i,j_k}$ for all integers i and k between 1 and m.

We let $t_{i,j} = (M_B^{-1}A)_{i,j}$ and $s_i = (M_B^{-1}\mathbf{b})_i$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)}$$

for $j = 1, 2, \ldots, n$, and

$$\mathbf{b} = \sum_{i=1}^{m} s_i\mathbf{a}^{(j_i)}.$$

A basis B determines an associated basic feasible solution if and only if $s_i \geq 0$ for $i = 1, 2, \ldots, m$. We suppose in what follows that the basis B determines a basic feasible solution.

Let

$$C = \sum_{i=1}^{m} c_{j_i} s_i.$$

Then C is the cost of the basic feasible solution associated with the basis B.

Let

$$-q_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j.$$

Then $q_j = 0$ for all $j \in \{j_1, j_2, \ldots, j_m\}$. Also the cost of any feasible solution $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ of the linear programming problem is

$$C + \sum_{j=1}^{n} q_j \overline{x}_j.$$

4. The Simplex Method (continued)

The *simplex tableau* associated with the basis B is that portion of the extended simplex tableau that omits the columns labelled by $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$. The simplex table has the following structure:

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C

4. The Simplex Method (continued)

Let $\mathbf{c}_B$ denote the m-dimensional vector defined such that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{pmatrix}.$$

Then the simplex tableau can be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$
	$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$

Example

We consider again the following linear programming problem:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:

$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11;$
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6;$
$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5.$

We are given the following initial basic feasible solution $(1, 2, 0, 0, 0)$. We need to determine whether this initial basic feasible solution is optimal and, if not, how to improve it till we obtain an optimal solution.

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 that maximizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix},$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

Our initial basis B satisfies $B = \{j_1, j_2\}$, where $j_1 = 1$ and $j_2 = 2$. The first two columns of the matrix A provide the corresponding invertible 2×2 matrix M_B. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

Inverting this matrix, we find that

$$M_B^{-1} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

For each integer j between 1 and 5, let $\mathbf{a}^{(j)}$ denote the m-dimensional vector whose ith component is $A_{i,j}$ for $i = 1, 2$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{2} t_{i,j} \mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{2} s_i \mathbf{a}^{(j_i)},$$

where $t_{i,j} = (M_B^{-1} A)_{i,j}$ and $s_i = (M_B^{-1} \mathbf{b})_i$ for $j = 1, 2, 3, 4, 5$ and $i = 1, 2$.

Calculating $M_B^{-1}A$ we find that

$$M_B^{-1}A = \begin{pmatrix} 1 & 0 & \frac{5}{7} & \frac{17}{7} & \frac{9}{7} \\ 0 & 1 & \frac{1}{7} & -\frac{12}{7} & -\frac{8}{7} \end{pmatrix}.$$

Also

$$M_B^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The coefficients of these matrices determine the values of $t_{i,j}$ and s_i to be entered into the appropriate cells of the simplex tableau.

The basis rows of the simplex tableau corresponding to the basis $\{1, 2\}$ are thus as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(1)}$	1	0	$\frac{5}{7}$	$\frac{17}{7}$	$\frac{9}{7}$	1
$\mathbf{a}^{(2)}$	0	1	$\frac{1}{7}$	$-\frac{12}{7}$	$-\frac{8}{7}$	2
	.	.	.	.	.	.

Now the cost C of the current feasible solution satisfies the equation

$$C = \sum_{i=1}^{2} c_{j_i} s_i = c_1 s_1 + c_2 s_2,$$

where $c_1 = 3$, $c_2 = 4$, $s_1 = 1$ and $s_2 = 2$. It follows that $C = 11$.

To complete the simplex tableau, we need to compute $-q_j$ for $j = 1, 2, 3, 4, 5$, where

$$-q_j = \sum_{i=1}^{2} c_{j_i} t_{i,j} - c_j.$$

Let $\mathbf{c}_B$ denote the 2-dimensional vector whose ith component is (c_{j_i}). Then $\mathbf{c}_B = (3, 4)$. Let $\mathbf{q}$ denote the 5-dimensional vector whose jth component is q_j for $j = 1, 2, 3, 4, 5$. Then

$$-\mathbf{q}^T = \mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T.$$

It follows that

$$\begin{aligned} -\mathbf{q}^T &= \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{5}{7} & \frac{17}{7} & \frac{9}{7} \\ 0 & 1 & \frac{1}{7} & -\frac{12}{7} & -\frac{8}{7} \end{pmatrix} \\ &\qquad - \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \frac{5}{7} & -\frac{60}{7} & -\frac{40}{7} \end{pmatrix}. \end{aligned}$$

The simplex tableau corresponding to basis $\{1, 2\}$ is therefore completed as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(1)}$	1	0	$\frac{5}{7}$	$\frac{17}{7}$	$\frac{9}{7}$	1
$\mathbf{a}^{(2)}$	0	1	$\frac{1}{7}$	$-\frac{12}{7}$	$-\frac{8}{7}$	2
	0	0	$\frac{5}{7}$	$-\frac{60}{7}$	$-\frac{40}{7}$	11

The values of $-q_j$ for $j = 1, 2, 3, 4, 5$ are not all non-positive ensures that the initial basic feasible solution is not optimal. Indeed the cost of a feasible solution $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$ is

$$11 - \tfrac{5}{7}\overline{x}_3 + \tfrac{60}{7}\overline{x}_4 + \tfrac{40}{7}\overline{x}_5.$$

Thus a feasible solution with $\overline{x}_3 > 0$ and $\overline{x}_4 = \overline{x}_5 = 0$ will have lower cost than the initial feasible basic solution. We therefore implement a change of basis whose pivot column is that labelled by the vector $\mathbf{a}^{(3)}$.

We must determine which row to use as the pivot row. We need to determine the value of i that minimizes the ratio $\dfrac{s_i}{t_{i,3}}$, subject to the requirement that $t_{i,3} > 0$. This ratio has the value $\frac{7}{5}$ when $i = 1$ and 14 when $i = 2$. Therefore the pivot row is the row labelled by $\mathbf{a}^{(1)}$. The pivot element $t_{1,3}$ then has the value $\frac{5}{7}$.

The simplex tableau corresponding to basis $\{2, 3\}$ is then obtained by subtracting the pivot row multiplied by $\frac{1}{5}$ from the row labelled by $\mathbf{a}^{(2)}$, subtracting the pivot row from the criterion row, and finally dividing all values in the pivot row by the pivot element $\frac{5}{7}$.

The simplex tableau for the basis $\{2, 3\}$ is thus the following:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(3)}$	$\frac{7}{5}$	0	1	$\frac{17}{5}$	$\frac{9}{5}$	$\frac{7}{5}$
$\mathbf{a}^{(2)}$	$-\frac{1}{5}$	1	0	$-\frac{11}{5}$	$-\frac{7}{5}$	$\frac{9}{5}$
	-1	0	0	-11	-7	10

All the values in the criterion row to the left of the new cost are non-positive. It follows that we have found a basic optimal solution to the linear programming problem. The values recorded in the column labelled by **b** show that this basic optimal solution is

$$(0, \tfrac{9}{5}, \tfrac{7}{5}, 0, 0).$$

4.9. The Revised Simplex Algorithm

The Simplex Tableau Algorithm restricts attention to the columns to the left of the extended simplex tableau. The Revised Simplex Algorithm proceeds by maintaining the columns to the right of the extended simplex tableau, calculating values in the columns to the left of that tableau only as required.

We show how the Revised Simplex Algorithm is implemented by applying it to the example used to demonstrate the implementation of the Simplex Algorithm.

Example

We apply the Revised Simplex Algorithm to determine a basic optimal solution to the the following linear programming problem:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:

$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11;$
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6;$
$x_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$

We are given the following initial basic feasible solution $(1, 2, 0, 0, 0)$. We need to determine whether this initial basic feasible solution is optimal and, if not, how to improve it till we obtain an optimal solution.

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 that maximizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix},$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

Our initial basis B satisfies $B = \{j_1, j_2\}$, where $j_1 = 1$ and $j_2 = 2$. The first two columns of the matrix A provide the corresponding invertible 2×2 matrix M_B. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

Inverting this matrix, we find that

$$M_B^{-1} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

For each integer j between 1 and 5, let $\mathbf{a}^((j)$ denote the m-dimensional vector whose ith component is $A_{i,j}$ for $i = 1, 2$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{2} t_{i,j} \mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{2} s_i \mathbf{a}^{(j_i)},$$

where $t_{i,j} = (M_B^{-1} A)_{i,j}$ and $s_i = (M_B^{-1} \mathbf{b})_i$ for $j = 1, 2, 3, 4, 5$ and $i = 1, 2$.

Let $r_{i,k} = (M_B^{-1})_{i,k}$ for $i = 1, 2$ and $k = 1, 2$, and let

$$\begin{aligned} C &= c_{j_1} s_1 + c_{j_2} s_2 = c_1 s_1 + c_2 s_2 = 11 \\ p_1 &= c_{j_1} r_{1,1} + c_{j_2} r_{2,1} = c_1 r_{1,1} + c_2 r_{2,1} = \frac{13}{7} \\ p_2 &= c_{j_1} r_{1,2} + c_{j_2} r_{2,2} = c_1 r_{1,2} + c_2 r_{2,2} = -\frac{11}{7} \end{aligned}$$

The values of s_i, $r_{i,k}$, C and p_k are inserted into the following tableau, which consists of the columns to the right of the extended simplex tableau:—

	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(1)}$	1	$-\frac{1}{7}$	$\frac{3}{7}$
$\mathbf{a}^{(2)}$	2	$\frac{4}{7}$	$-\frac{5}{7}$
	11	$\frac{13}{7}$	$-\frac{11}{7}$

To proceed with the algorithm, one computes values $-q_j$ for $j \notin B$ using the formula

$$-q_j = p_1 A_{1,j} + p_2 A_{2,j} - c_j,$$

seeking a value of j for which $-q_j > 0$. Were all the values $-q_j$ are non-positive (i.e., if all the q_j are non-negative), then the initial solution would be optimal. Computing $-q_j$ for $j = 5, 4, 3$, we find that

$$\begin{array}{rcl} -q_5 & = & \frac{13}{7} \times 3 - \frac{11}{7} \times 4 - 5 = -\frac{40}{7} \\ -q_4 & = & = \frac{13}{7} \times 7 - \frac{11}{7} \times 8 - 9 = -\frac{60}{7} \\ -q_3 & = & = \frac{13}{7} \times 4 - \frac{11}{7} \times 3 - 2 = \frac{5}{7} \end{array}$$

The inequality $q_3 > 0$ shows that the initial basic feasible solution is not optimal, and we should seek to change basis so as to include the vector $\mathbf{a}^{(3)}$. Let

$$\begin{aligned} t_{1,3} &= r_{1,1}A_{1,3} + r_{1,2}A_{2,3} = -\tfrac{1}{7} \times 4 + \tfrac{3}{7} \times 3 = \tfrac{5}{7} \\ t_{2,3} &= r_{2,1}A_{1,3} + r_{2,2}A_{2,3} = \tfrac{4}{7} \times 4 - \tfrac{5}{7} \times 3 = \tfrac{1}{7} \end{aligned}$$

Then

$$\mathbf{a}^{(3)} = t_{1,3}\mathbf{a}^{(j_1)} + t_{2,3}\mathbf{a}^{(j_2)} = \tfrac{5}{7}\mathbf{a}^{(1)} + \tfrac{1}{7}\mathbf{a}^{(2)}.$$

We introduce a column representing the vector $\mathbf{a}^{(3)}$ into the tableau to serve as a pivot column. The resultant tableau is as follows:—

	$\mathbf{a}^{(3)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(1)}$	$\frac{5}{7}$	1	$-\frac{1}{7}$	$\frac{3}{7}$
$\mathbf{a}^{(2)}$	$\frac{1}{7}$	2	$\frac{4}{7}$	$-\frac{5}{7}$
	$\frac{5}{7}$	11	$\frac{13}{7}$	$-\frac{11}{7}$

To determine a pivot row we must pick the row index i so as to minimize the ratio $\dfrac{s_i}{t_{i,3}}$, subject to the requirement that $t_{i,3} > 0$. In the context of this example, we should pick $i = 1$. Accordingly the row labelled by the vector $\mathbf{a}^{(1)}$ is the pivot row. To implement the change of basis we must subtract from the second row the values above them in the pivot row, multiplied by $\frac{1}{5}$; we must subtract the values in the pivot row from the values below them in the criterion row, and we must divide the values in the pivot row itself by the pivot element $\frac{5}{7}$.

The resultant tableau corresponding to the basis $2, 3$ is then as follows:—

	$\mathbf{a}^{(3)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(3)}$	1	$\frac{7}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$
$\mathbf{a}^{(2)}$	0	$\frac{9}{5}$	$\frac{3}{5}$	$-\frac{4}{7}$
	0	10	2	-2

A straightforward computation then shows that if

$$\mathbf{p}^T = \begin{pmatrix} 2 & -2 \end{pmatrix}$$

then

$$\mathbf{p}^T A - \mathbf{c}^T = \begin{pmatrix} -1 & 0 & 0 & -11 & -7 \end{pmatrix}.$$

The components of this row vector are all non-positive. It follows that the basis $\{2, 3\}$ determines a basic optimal solution

$$(0, \tfrac{9}{5}, \tfrac{7}{5}, 0, 0).$$

4.10. Finding Initial Basic Solutions

Suppose that we are given a linear programming problem in Dantzig standard form, specified by positive integers m and n, an $m \times n$ matrix A of rank m, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. The problem requires us to find an n-dimensional vector $\mathbf{x}$ that minimizes the objective function $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Now, in the event that the column vector $\mathbf{b}$ has negative coefficients, the relevant rows of the constraint matrix A and target vector $\mathbf{b}$ can be multiplied by -1 to yield an equivalent problem in which the coefficients of the target vector are all non-negative. Therefore we may assume, without loss of generality, that $\mathbf{b} \geq \mathbf{0}$.

The Simplex Tableau Algorithm and the Revised Simplex Algorithm provided methods for passing from an initial basic feasible solution to a basic optimal solution, provided that such a basic optimal solution exists. However, we need first to find an initial basic feasible solution for this linear programming problem.

One can find such an initial basic feasible solution by solving an auxiliary linear programming problem. This auxiliary problem requires us to find n-dimensional vectors $\mathbf{x}$ and $\mathbf{z}$ that minimize the objective function $\sum_{j=1}^{n} (\mathbf{z})_j$ subject to the constraints $A\mathbf{x} + \mathbf{z} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$.

This auxiliary linear programming problem is itself in Dantzig standard form. Moreover it has an initial basic feasible solution specified by the simultaneous equations $\mathbf{x} = \mathbf{0}$ and $\mathbf{z} = \mathbf{b}$. The objective function of a feasible solution is always non-negative. Applications of algorithms based on the Simplex Method should identify a basic optimal solution $(\mathbf{x}, \mathbf{z})$ for this problem. If the cost $\sum_{j=1}^{n} (\mathbf{z})_j$ of this basic optimal solution is equal to zero then $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. If the cost of the basic optimal solution is positive then the problem does not have any basic feasible solutions.

The process of solving a linear programming problem in Dantzig standard form thus typically consists of two *phases*. The *Phase I* calculation aims to solve the auxiliary linear programming problem of seeking n-dimensional vectors $\mathbf{x}$ and $\mathbf{z}$ that minimize $\sum_{i=1}^{n} (\mathbf{z})_j$ subject to the constraints $A\mathbf{x} + \mathbf{z} = \mathbf{b}$, $\mathbf{x} \geq 0$ and $\mathbf{z} \geq 0$. If the optimal solution $(\mathbf{x}, \mathbf{z})$ of the auxiliary problem satisfies $\mathbf{z} \neq \mathbf{0}$ then there is no initial basic solution of the original linear programming problem. But if $\mathbf{z} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and thus the Phase I calculation has identified an initial basic feasible solution of the original linear programmming problem. The *Phase II* calculation is the process of successively changing bases to lower the cost of the corresponding basic feasible solutions until either a basic optimal solution has been found or else it has been demonstated that no such basic optimal solution exists.