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David R. Wilkins

Given an $m \times n$ matrix A of rank m, an m-dimensional target vector $\mathbf{b}$, and an n-dimensional cost vector $\mathbf{c}$, there exists an extended simplex tableau associated with any basis B for the linear programming problem, irrespective of whether or not there exists a basic feasible solution associated with the given basis B.

The crucial requirement that enables the construction of the tableau is that the basis B should consist of m distinct integers $j_1, j_2, \ldots, j_m$ between 1 and m for which the corresponding columns of the matrix A constitute a basis of the vector space $\mathbb{R}^m$.

A basis B is associated with a basic feasible solution of the linear programming problem if and only if the values in the column labelled by the target vector $\mathbf{b}$ and the rows labelled by $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ should be non-negative. If so, those values will include the non-zero components of the basic feasible solution associated with the basis.

If there exists a basic feasible solution associated with the basis B then that solution is optimal if and only if all the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ are all non-positive.

Versions of the Simplex Tableau Algorithm for determining a basic optimal solution to the linear programmming problem, given an initial basic feasible solution, rely on the transformation rules that determine how the values in the body of the extended simplex tableau are transformed on passing from an old basis B to an new basis B', where the new basis B' contains all but one of the members of the old basis B.

Let us refer to the rows of the extended simplex tableau labelled by the basis vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ as the *basis rows* of the tableau. The following lemma determines how elements of the basis rows of the tableau transform under changes of column bases that replace a single column of an initial basis by another column that is linearly independent of the remaining columns of that initial basis.

Lemma 4.4

Let A be an $m \times n$ matrix of rank m with real coefficients, let $j_1, j_2, \ldots, j_m$ be distinct integers between 1 and n, let h be an integer between 1 and m, and let $j'_1, j'_2, \ldots, j'_m$ be distinct integers between 1 and n, where $j'_h \neq j_h$ and $j_i = j'_i$ for $i \neq h$. Suppose that the column vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ are linearly independent, and that the column vectors $\mathbf{a}^{(j'_1)}, \mathbf{a}^{(j'_2)}, \ldots, \mathbf{a}^{(j'_m)}$ are also linearly independent, where $\mathbf{a}^{(j)}$ denotes the jth column of the matrix A. Let $\mathbf{v}$ be an element of $\mathbb{R}^m$, let $z_1, z_2, \ldots, z_m$, $z'_1, z'_2, \ldots, z'_m$, $t_{1,j'_h}, t_{2,j'_h}, \ldots, t_{m,j'_h}$ denote the uniquely-determined real numbers for which

$$\mathbf{v} = \sum_{i=1}^{m} z_i \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} z'_i \mathbf{a}^{(j'_i)}$$

and

$$\mathbf{a}^{(j'_h)} = \sum_{i=1}^{m} t_{i,j'_h} \mathbf{a}^{(j_i)}.$$

Then

$$z'_h = \frac{1}{t_{h,j'_h}} z_h$$

and

$$z'_i = z_i - \frac{t_{i,j'_h}}{t_{h,j'_h}} z_h \quad (i \neq h).$$

Proof

Expressing the vector $\mathbf{v}$ as a linear combination of $\mathbf{a}^{(j'_h)}$ and the vectors $\mathbf{a}^{(j_i)}$ for $i \neq j$, and then substituting in the representation of $\mathbf{a}^{(j'_h)}$ as a linear combination of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_m)}$, and using the requirement that $j'_i = j_i$ when $i \neq h$, we find that

$$
\begin{aligned}
\mathbf{v} &= \sum_{i=1}^{m} z_i' \mathbf{a}^{(j_i')} \\
&= z_h' \mathbf{a}^{(j_h')} + \sum_{\substack{1 \le i \le m \\ i \ne h}} z_i' \mathbf{a}^{(j_i)} \\
&= z_h' t_{h,j_h'} \mathbf{a}^{(j_h)} + \sum_{\substack{1 \le i \le m \\ i \ne h}} (z_i' + z_h' t_{i,j_h'}) \mathbf{a}^{(j_i)}.
\end{aligned}
$$

Equating coefficients of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots \mathbf{a}^{(j_m)}$, we deduce that

$$z_h = z_h' t_{h,j_h'}$$

and

$$z_i = z_i' + z_h' t_{i,j_h'} \qquad (1 \le i \le m \text{ and } i \ne h).$$

It follows that

$$z'_h = \frac{1}{t_{h,j'_h}} z_h$$

and

$$z'_i = z_i - \frac{t_{i,j'_h}}{t_{h,j'_h}} z_h \quad (i \neq h),$$

as required. ∎

We now apply Lemma 4.4 in order to determine how entries in the basis rows of the extended simplex tableau transform which one element of the basis is replaced by an element not belonging to the basis.

Thus we consider the manner in which the basis rows of the extended simplex tableau transform under such a change of basis. Let A be be $m \times n$ matrix of rank m and let $\mathbf{b}$ be the m-dimensional target vector that are employed in the specification of the linear programming problem. Let the old basis B consist of distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n, and let the new basis B' also consist of distinct integers $j'_1, j'_2, \ldots, j'_m$ between 1 and n. We suppose that the new basis B' is obtained from the old basis by replacing an element j_h of the old basis B by some integer j'_h between 1 and n that does not belong to the old basis. We suppose therefore that $j_i = j'_i$ when $i \neq h$, and that j'_h is some integer between 1 and n that does not belong to the basis B.

Let the coefficients $t_{i,j}$, $t'_{i,j}$, s_i, s'_i, $r_{i,k}$ and $r'_{i,k}$ be determined for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j}\mathbf{a}^{(j'_i)}$$

for $j = 1, 2, \ldots, n$,

$$\mathbf{b} = \sum_{i=1}^{m} s_i\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s'_i\mathbf{a}^{(j'_i)}$$

and

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} r'_{i,k}\mathbf{a}^{(j'_i)}$$

for $k = 1, 2, \ldots, m$.

It then follows from direct applications of Lemma 4.4 that

$$
\begin{aligned}
t'_{h,j} &= \frac{1}{t_{h,j'_h}}\, t_{h,j}, \\
t'_{i,j} &= t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, t_{h,j} \quad (i \neq h). \\
s'_h &= \frac{1}{t_{h,j'_h}}\, s_h, \\
s'_i &= s_i - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, s_h \quad (i \neq h), \\
r'_{h,k} &= \frac{1}{t_{h,j'_h}}\, r_{h,k}, \\
r'_{i,k} &= r_{i,k} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, r_{h,k} \quad (i \neq h).
\end{aligned}
$$

The *pivot row* of the extended simplex tableau for this change of basis from B to B' is the row labelled by the basis vector $\mathbf{a}^{(j_h)}$ that is to be removed from the current basis. The *pivot column* of the extended simplex tableau for this change of basis is the column labelled by the vector $\mathbf{a}^{(j'_h)}$ that is to be added to the current basis. The *pivot element* for this change of basis is the element t_{h,j'_h} of the tableau located in the pivot row and pivot column of the tableau.

The identities relating the components of $\mathbf{a}^{(j)}$, $\mathbf{b}$ and $\mathbf{e}^{(k)}$ with respect to the old basis to the components of those vectors with respect to the new basis ensure that the rules for transforming the rows of the tableau other than the criterion row can be stated as follows:—

- a value recorded in the pivot row is transformed by dividing it by the pivot element;
- an value recorded in a basis row other than the pivot row is transformed by substracting from it a constant multiple of the value in the same column that is located in the pivot row, where this constant multiple is the ratio of the values in the basis row and pivot row located in the pivot column.

In order to complete the discussion of the rules for transforming the values recorded in the extended simplex tableau under a change of basis that replaces an element of the old basis by an element not in that basis, it remains to analyse the rule that determines how the elements of the criterion row are transformed under this change of basis.

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, \ldots, n$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j}\mathbf{a}^{(j'_i)},$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$. Moreover

$$t'_{h,j} = \frac{1}{t_{h,j'_h}}\, t_{h,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, t_{h,j}$$

for all integers i between 1 and m for which $i \neq h$.

Now

$$-q_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j \quad \text{and} \quad -q'_j = \sum_{i=1}^{m} c_{j'_i} t'_{i,j} - c_j.$$

Therefore

$$\begin{aligned} q_j - q'_j &= \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_{j_i}(t'_{i,j} - t_{i,j}) + c_{j'_h} t'_{h,j} - c_{j_h} t_{h,j} \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) t_{h,j} \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} t_{h,j} \end{aligned}$$

and thus

$$-q'_j = -q_j + \frac{q_{j'_h}}{t_{h,j'_h}} t_{h,j}$$

for $j = 1, 2, \ldots, n$.

Next we note that

$$C = \sum_{i=1}^{m} c_{j_i} s_i \quad \text{and} \quad C' = \sum_{i=1}^{m} c_{j'_i} s'_i.$$

Therefore

$$\begin{aligned} C' - C &= \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_{j_i}(s'_i - s_i) + c_{j'_h} s'_h - c_{j_h} s_h \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) s_h \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} s_h \end{aligned}$$

and thus

$$C' = C + \frac{q_{j'_h}}{t_{h,j'_h}} s_h$$

for $k = 1, 2, \ldots, m$.

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k} \quad \text{and} \quad p'_k = \sum_{i=1}^{m} c_{j'_i} r'_{i,k}.$$

Therefore

$$\begin{aligned} p'_k - p_k &= \sum_{\substack{1 \le i \le m \\ i \ne h}} c_{j_i}(r'_{i,k} - r_{i,k}) + c_{j'_h} r'_{h,k} - c_{j_h} r_{h,k} \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) r_{h,k} = \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k} \end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k}$$

for $k = 1, 2, \ldots, m$.

We conclude that the criterion row of the extended simplex tableau transforms under changes of basis that replace one element of the basis according to a rule analogous to that which applies to the basis rows. Indeed an element of the criterion row is transformed by subtracting from it a constant multiple of the element in the pivot row that belongs to the same column, where the multiplying factor is the ratio of the elements in the criterion row and pivot row of the pivot column.

We have now discussed how the extended simplex tableau associated with a given basis B is constructed from the constraint matrix A, target vector $\mathbf{b}$ and cost vector $\mathbf{c}$ that characterizes the linear programming problem. We have also discussed how the tableau transforms when one element of the given basis is replaced.

It remains how to replace an element of a basis associated with a non-optimal feasible solution so as to obtain a basic feasible solution of lower cost where this is possible.

We use the notation previously established. Let $j_1, j_2, \ldots, j_m$ be the elements of a basis B that is associated with some basic feasible solution of the linear programming problem. Then there are non-negative numbers $s_1, s_2, \ldots, s_m$ such that

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)},$$

where $\mathbf{a}^{(j_i)}$ is the m-dimensional vector determined by column j_i of the constraint matrix A.

Let j_0 be an integer between 1 and n that does not belong to the basis B. Then

$$\mathbf{a}^{(j_0)} - \sum_{i=1}^{m} t_{i,j_0} \mathbf{a}^{(j_i)} = \mathbf{0}.$$

and therefore

$$\lambda \mathbf{a}^{(j_0)} + \sum_{i=1}^{m} (s_i - \lambda t_{i,j_0}) \mathbf{a}^{(j_i)} = \mathbf{b}.$$

This expression representing $\mathbf{b}$ as a linear combination of the basis vectors $\mathbf{a}^{(j_0)}, \mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ determines an n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ satisfying the matrix equation $A\overline{\mathbf{x}}(\lambda) = \mathbf{b}$. Let $\overline{x}_j(\lambda)$ denote the jth component of the vector $\overline{\mathbf{x}}(\lambda)$ for $j = 1, 2, \ldots, n$. Then

- $\overline{x}_{j_0}(\lambda) = \lambda$;
- $\overline{x}_{j_i}(\lambda) = s_i - \lambda t_{i,j_0}$ for $i = 1, 2, \ldots, m$;
- $\overline{x}_j = 0$ when $j \notin \{j_0, j_1, j_2, \ldots, j_m\}$.

The n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ represents a feasible solution of the linear programming problem if and only if all its coefficients are non-negative. The cost is then $C + q_{j_0}\lambda$, where C is the cost of the basic feasible solution determined by the basis B.

Suppose that $q_{j_0} < 0$ and that $t_{i,j_0} \leq 0$ for $i = 1, 2, \ldots, m$. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution with cost $C + q_{j_0}\lambda$ for all non-negative real numbers λ. In this situation there is no optimal solution to the linear programming problem, because, given any real number K, it is possible to choose λ so that $C + q_{j_0}\lambda < K$, thereby obtaining a feasible solution whose cost is less than K.

If there does exist an optimal solution to the linear programming problem then there must exist at least one integer i between 1 and m for which $t_{i,j_0} > 0$. We suppose that this is the case. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution if and only if λ satisfies $0 \leq \lambda \leq \lambda_0$, where

$$\lambda_0 = \text{minimum}\left(\frac{s_i}{t_{i,j_0}} : t_{i,j_0} > 0\right).$$

We can then choose some integer h between 1 and n for which

$$\frac{s_h}{t_{h,j_0}} = \lambda_0.$$

Let $j'_i = j_i$ for $i \neq h$, and let $j'_h = j_0$, and let $B' = \{j'_1, j'_2, \ldots, j'_m\}$. Then $\overline{\mathbf{x}}(\lambda_0)$ is a basic feasible solution of the linear programming problem associated with the basis B'. The cost of this basic feasible solution is

$$C + \frac{s_h q_{j_0}}{t_{h,j_0}}.$$

It makes sense to select the replacement column so as to obtain the greatest cost reduction. The procedure for finding this information from the tableau can be described as follows.

We suppose that the simplex tableau for a basic feasible solution has been prepared. Examine the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$. If all those are non-positive then the basic feasible solution is optimal. If not, then consider in turn those columns $\mathbf{a}^{(j_0)}$ for which the value $-q_{j_0}$ in the criterion row is positive. For each of these columns, examine the coefficients recorded in the column in the basis rows. If these coefficients are all non-positive then there is no optimal solution to the linear programming problem. Otherwise choose h to be the value of i that minimizes the ratio $\dfrac{s_i}{t_{i,j_0}}$ amongst those values of i for which $t_{i,j_0} > 0$. The row labelled by $\mathbf{a}^{(j_h)}$ would then be the pivot row if the column $\mathbf{a}^{(j_0)}$ were used as the pivot column.

Calculate the value of the cost reduction $\dfrac{s_h(-q_{j_0})}{t_{h,j_0}}$ that would result if the column labelled by $\mathbf{a}^{(j_0)}$ were used as the pivot column. Then choose the pivot column to maximize the cost reduction amongst all columns $\mathbf{a}^{(j_0)}$ for which $-q_{j_0} > 0$. Choose the row labelled by $\mathbf{a}^{(j_h)}$, where h is determined as described above. Then apply the procedures for transforming the simplex tableau to that determined by the new basis B', where B' includes j_0 together with j_i for all integers i between 1 and m satisfying $i \neq h$.