MA3484—Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 16 (February 23, 2017)

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4.7. The Extended Simplex Tableau

We now consider the construction of a tableau for a linear programming problem in Dantzig standard form. Such a problem is specified by an $m \times n$ matrix A, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. We suppose moreover that the matrix A is of rank m. We consider procedures for solving the following linear program in Danzig standard form.

> *Determine $\mathbf{x} \in \mathbb{R}^n$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

We denote by $A_{i,j}$ the component of the matrix A in the ith row and jth column, we denote by b_i the ith component of the target vector $\mathbf{b}$ for $i = 1, 2, \ldots, m$, and we denote by c_j the jth component of the cost vector $\mathbf{c}$ for $j = 1, 2, \ldots, n$.

We recall that a feasible solution to this problem consists of an n-dimensional vector $\mathbf{x}$ that satisfies the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ (see Subsection 2). A feasible solution of the linear programming problem then consists of non-negative real numbers $x_1, x_2, \ldots, x_n$ for which

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \mathbf{b}.$$

A feasible solution determined by $x_1, x_2, \ldots, x_n$ is optimal if it minimizes cost $\sum_{j=1}^{n} c_j x_j$ amongst all feasible solutions to the linear programming problem.

Let $j_1, j_2, \ldots, j_m$ be distinct integers between 1 and n that are the elements of a basis B for the linear programming problem. Then the vectors $\mathbf{a}^{(j)}$ for $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$. (see Subsection 4).

We denote by M_B the invertible $m \times m$ matrix whose component $(M)_{i,k}$ in the ith row and jth column satisfies $(M_B)_{i,k} = (A)_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then the kth column of the matrix M_B is specified by the column vector $\mathbf{a}^{(j_k)}$ for $k = 1, 2, \ldots, m$, and thus the columns of the matrix M_B coincide with those columns of the matrix A that are determined by elements of the basis B.

Proposition 4.3

Let A be an $m \times n$ matrix with real coefficients that is of rank m whose columns are represented by the column vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$, let $\mathbf{b}$ be an m-dimensional column vector, and let $B = \{j_1, j_2, \ldots, j_m\}$, where $j_1, j_2, \ldots, j_m$ are integers between 1 and n for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. Let M_B be the invertible $m \times m$ matrix defined so that $(M_B)_{i,k} = A_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then there are uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ for which

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} \quad \textit{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

Moreover

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $j = 1, 2, \ldots, n$, where $r_{i,k} = (M_B^{-1})_{i,k}$ for $i, k = 1, 2, \ldots, m$.

Proof

Every vector in $\mathbb{R}^m$ can be expressed as a linear combination of the basis vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$. It follows that there exist uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

Then

$$A_{i,j} = \sum_{k=1}^{n} t_{k,j} A_{i,j_k} = \sum_{k=1}^{n} (M_B)_{i,k} t_{k,j}$$

and

$$b_i = \sum_{k=1}^{m} s_k A_{i,j_k} = \sum_{k=1}^{n} (M_B)_{i,k} s_k.$$

Thus $\mathbf{a}^{(j)} = M_B \mathbf{t}^{(j)}$ and $\mathbf{b} = M_B \mathbf{s}$ for $j = 1, 2, \ldots, n$, where $\mathbf{t}^{(j)}$ and $\mathbf{s}$ denote the column vectors that satisfy $(\mathbf{t}^{(j)})_i = t_{i,j}$ and $(\mathbf{s})_i = s_i$ for $i = 1, 2, \ldots, m$. It follows that

$$\mathbf{t}^{(j)} = M_B^{-1} \mathbf{a}^{(j)} \quad \text{and} \quad \mathbf{s} = M_B^{-1} \mathbf{b}$$

for $j = 1, 2, \ldots, n$. Thus

$$t_{i,j} = (M_B^{-1} \mathbf{a}^{(j)})_i = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = (M_B^{-1} \mathbf{b})_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $i = 1, 2, \ldots, m$, where $r_{i,k} = (M_B^{-1})_{i,k}$ for $i, k = 1, 2, \ldots, m$. This completes the proof. ∎

Let A be an $m \times n$ matrix with real coefficients that is of rank m whose columns are represented by the column vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$, and let $B = \{j_1, j_2, \ldots, j_m\}$, where $j_1, j_2, \ldots, j_m$ are integers between 1 and n for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. Let M_B be the invertible $m \times m$ matrix defined so that $(M_B)_{i,k} = A_{i,j_k}$ for $i, k = 1, 2, \ldots, m$.

The standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$ is defined such that $(\mathbf{e}^{(k)})_i = \delta_{i,k}$ for $i, k = 1, 2, \ldots, m$, where $\delta_{i,k}$ is the *Kronecker delta*, defined such that

$$\delta_{i,k} = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

It follows from Proposition 4.3 (with the column vector **b** of that proposition set equal to $\mathbf{e}^{(k)}$) that

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} \sum_{h=1}^{m} r_{i,h} (\mathbf{e}^{(k)})_h \mathbf{a}^{(i)} = \sum_{i=1}^{m} r_{i,k} \mathbf{a}^{(i)},$$

where $r_{i,k}$ is the coefficient $(M_B^{-1})_{i,k}$ in the ith row and kth column of the inverse M_B^{-1} of the matrix M_B.

Let A be an $m \times n$ matrix of rank m with real coefficients, and let $\mathbf{b}$ be an m-dimensional vector, and let $\{j_1, j_2, \ldots, j_m\}$ be a subset of $\{1, 2, \ldots, n\}$ for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. We can then record the coefficients of the m-dimensional vectors

$$\mathbf{a}^{(1)},\ \mathbf{a}^{(2)}, \ldots,\ \mathbf{a}^{(n)},\ \mathbf{b},\ \mathbf{e}^{(1)},\ \mathbf{e}^{(2)}, \ldots,\ \mathbf{e}^{(m)}$$

with respect to the basis $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$, of $\mathbb{R}^m$ in a tableau of the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

The definition of the quantities $t_{i,j}$ ensures that $t_{i,j_k} = \delta_{i,k}$ for $i = 1, 2, \ldots, m$, where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

Also it follows from Proposition 4.3 that

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{i,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $i = 1, 2, \ldots, m$.

If the quantities $s_1, s_2, \ldots, s_m$ are all non-negative then they determine a basic feasible solution $\mathbf{x}$ of the linear programming problem associated with the basis B with components $x_1, x_2, \ldots, x_n$, where $x_{j_i} = s_i$ for $i = 1, 2, \ldots, m$ and $x_j = 0$ for all integers j between 1 and n that do not belong to the basis B. Indeed

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \sum_{i=1}^{m} x_{j_i} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

The *cost* C of the basic feasible solution $\mathbf{x}$ is defined to be the value $\overline{c}^T\mathbf{x}$ of the objective function. The definition of the quantities $s_1, s_2, \ldots, s_m$ ensures that

$$C = \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} c_{j_i} s_i.$$

If the quantities $s_1, s_2, \ldots, s_n$ are not all non-negative then there is no basic feasible solution associated with the basis B.

The criterion row at the bottom of the tableau has cells to record quantities $p_1, p_2, \ldots, p_m$ associated with the vectors that constitute the standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$. These quantities are defined so that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$, where c_{j_i} is the cost associated with the basis vector $\mathbf{a}^{(j_i)}$ for $i = 1, 2, \ldots, k$, Now the quantities $r_{i,k}$ are the components of the inverse of the matrix M_B, and therefore

$$\sum_{k=1}^{m} r_{h,k} A_{k,j_i} = \delta_{h,i}$$

for $h, i = 1, 2, \ldots, m$, where

$$\delta_{h,i} = \begin{cases} 1 & \text{if } h = i; \\ 0 & \text{if } h \neq i. \end{cases}$$

It follows that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = \sum_{k=1}^{m} \sum_{h=1}^{m} c_{j_h} r_{h,k} A_{k,j_i} = \sum_{h=1}^{m} c_{j_h} \left(\sum_{k=1}^{m} r_{h,k} A_{k,j_i} \right) = c_{j_i}$$

On combining the identities

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k, \quad p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k} \quad \text{and} \quad C = \sum_{i=1}^{m} c_{j_i} s_i$$

derived above, we find that

$$C = \sum_{i=1}^{m} c_{j_i} s_i = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{j_i} r_{i,k} b_k = \sum_{k=1}^{m} p_k b_k.$$

The tableau also has cells in the criterion row to record quantities

$$-q_1, -q_2, \ldots, -q_n,$$

where $q_1, q_2, \ldots, q_n$ are the components of the unique n-dimensional vector $\mathbf{q}$ characterized by the following properties:

- $q_{j_i} = 0$ for $i = 1, 2, \ldots, m$;
- $\mathbf{c}^T \overline{\mathbf{x}} = C + \mathbf{q}^T \overline{\mathbf{x}}$ for all $\overline{\mathbf{x}} \in \mathbb{R}^m$ satisfying the matrix equation $A\overline{\mathbf{x}} = \mathbf{b}$.

First we show that if $\mathbf{q} \in \mathbb{R}^n$ is defined such that $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ has the required properties.

The definition of $p_1, p_2, \ldots, p_k$ ensures (as noted above) that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = c_{j_i}$$

for $i = 1, 2, \ldots, k$. It follows that

$$q_{j_i} = c_{j_i} - (\mathbf{p}^T A)_{j_i} = c_{j_i} - \sum_{k=1}^{m} p_k A_{k,j_i} = 0$$

for $i = 1, 2, \ldots, n$.

Also $\mathbf{p}^T\mathbf{b} = C$. It follows that if $\overline{\mathbf{x}} \in \mathbb{R}^n$ satisfies $A\overline{\mathbf{x}} = \mathbf{b}$ then

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = C + \mathbf{q}^T\overline{\mathbf{x}}.$$

Thus if $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ satisfies the properties specified above.

We next show that

$$(\mathbf{p}^T A)_j = \sum_{i=1}^{m} c_{j_i} t_{i,j}$$

for $j = 1, 2, \ldots, n$.

Now

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. (see Proposition 4.3). Also the definition of p_k ensures that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$. These results ensure that

$$\sum_{i=1}^{m} c_{j_i} t_{i,j} = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{j_i} r_{i,k} A_{k,j} = \sum_{k=1}^{m} p_k A_{k,j} = (\mathbf{p}^T A)_j.$$

It follows that

$$-q_j = \sum_{k=1}^{m} p_k A_{k,j} - c_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j$$

for $j = 1, 2, \ldots, n$.

4. The Simplex Method (continued)

The *extended simplex tableau* associated with the basis B is obtained by entering the values of the quantities $-q_j$ (for $j = 1, 2, \ldots, n$), C and p_k (for $k = 1, 2, \ldots, m$) into the bottom row to complete the tableau described previously. The extended simplex tableau has the following structure:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C	p_1	p_2	$\cdots$	p_m

4. The Simplex Method (continued)

The extended simplex tableau can be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

4. The Simplex Method (continued)

Let $\mathbf{c}_B$ denote the m-dimensional vector defined so that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{pmatrix}.$$

The identities we have verified ensure that the extended simplex tableau can therefore also be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1} A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$