MA3484—Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 15 (February 17, 2017)

David R. Wilkins

We continue the discussion of how the extended simplex tableau transforms under a change of basis.

We now calculate the new values for the criterion row. The new basis B' is given by $B' = \{j'_1, j'_2, j'_3\}$, where $j'_1 = 1$, $j'_2 = 4$ and $j'_3 = 3$. The values p'_1, p'_2 and p'_3 that are to be recorded in the criterion row of the new tableau in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ respectively are determined by the equation

$$p'_k = c_{j'_1} r'_{1,k} + c_{j'_2} r'_{2,k} + c_{j'_3} r'_{3,k}$$

for $k = 1, 2, 3$, where

$$c_{j'_1} = c_1 = 2, \quad c_{j'_2} = c_4 = 1, \quad c_{j'_3} = c_3 = 3,$$

and where $r'_{i,k}$ denotes the ith component of the vector $\mathbf{e}^{(k)}$ with respect to the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(4)}, \mathbf{a}^{(3)}$ of $\mathbb{R}^3$.

We find that

$$\begin{aligned}
p_1' &= c_{j_1'} r_{1,1}' + c_{j_2'} r_{2,1}' + c_{j_3'} r_{3,1}' \\
&= 2 \times \left(-\tfrac{9}{27}\right) + 1 \times \tfrac{6}{27} + 3 \times \tfrac{6}{27} = \tfrac{6}{27}, \\
p_2' &= c_{j_1'} r_{1,2}' + c_{j_2'} r_{2,2}' + c_{j_3'} r_{3,2}' \\
&= 2 \times \tfrac{12}{27} + 1 \times \tfrac{7}{27} + 3 \times \left(-\tfrac{11}{27}\right) = -\tfrac{2}{27}, \\
p_3' &= c_{j_1'} r_{1,3}' + c_{j_2'} r_{2,3}' + c_{j_3'} r_{3,3}' \\
&= 2 \times \tfrac{3}{27} + 1 \times \left(-\tfrac{5}{27}\right) + 3 \times \tfrac{4}{27} = \tfrac{13}{27}.
\end{aligned}$$

We next calculate the cost C' of the new basic feasible solution. The quantities s_1', s_2' and s_3' satisfy $s_i' = x_{j_i}'$ for $i = 1, 2, 3$, where $(x_1', x_2', x_3', x_4', x_5')$ is the new basic feasible solution. It follows that

$$C' = c_{j_1'} s_1' + c_{j_2'} s_2' + c_{j_3'} s_3',$$

where s_1, s_2 and s_3 are determined so that

$$\mathbf{b} = s_1' \mathbf{a}^{(j_1')} + s_2' \mathbf{a}^{(j_2')} + s_3' \mathbf{a}^{(j_3')}.$$

The values of s_1', s_2' and s_3' have already been determined, and have been recorded in the column of the new tableau labelled by the vector $\mathbf{b}$.

We can therefore calculate C' as follows:—

$$\begin{aligned} C' &= c_{j_1'} s_1' + c_{j_2'} s_2' + c_{j_3'} s_3' = c_1 s_1' + c_4 s_2' + c_3 s_3' \\ &= 2 \times \frac{99}{27} + \frac{69}{27} + 3 \times \frac{15}{27} = \frac{312}{27}. \end{aligned}$$

Alternatively we can use the identity $C' = \mathbf{p}'^T \mathbf{b}$ to calculate C' as follows:

$$C' = p_1' b_1 + p_2' b_2 + p_3' b_3 = \frac{6}{27} \times 13 - \frac{2}{27} \times 13 + \frac{13}{27} \times 20 = \frac{312}{27}.$$

We now enter the values of p_1', p_2', p_3' and C' into the tableau associated with basis $\{1, 4, 3\}$. The tableau then takes the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

In order to complete the extended tableau, it remains to calculate the values $-q'_j$ for $j = 1, 2, 3, 4, 5$, where q'_j satisfies the equation $-q'_j = \mathbf{p}'^T \mathbf{a}_j - c_j$ for $j = 1, 2, 3, 4, 5$.

Now q'_j is the jth component of the vector $\mathbf{q}'$ that satisfies the matrix equation $-\mathbf{q}'^T = \mathbf{p}'^T A - \mathbf{c}^T$. It follows that

$$\begin{aligned}
-\mathbf{q}'^T &= \mathbf{p}'^T A - \mathbf{c}^T \\
&= \begin{pmatrix} \frac{6}{27} & \frac{-2}{27} & \frac{13}{27} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix} \\
&\qquad - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & \frac{32}{27} & 3 & 1 & \frac{76}{27} \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{76}{27} & 0 & 0 & -\frac{32}{27} \end{pmatrix}
\end{aligned}$$

Thus

$$q_1' = 0, \quad q_2' = \tfrac{76}{27}, \quad q_3' = 0, \quad q_4' = 0, \quad q_5' = \tfrac{32}{27}.$$

The value of each q_j' can also be calculated from the other values recorded in the column of the extended simplex tableau labelled by the vector $\mathbf{a}^{(j)}$. Indeed the vector $\mathbf{p}'$ is determined so as to satisfy the equation $\mathbf{p}'^T\mathbf{a}^{(j')} = c_{j'}$ for all $j' \in B'$. It follows that

$$\mathbf{p}'^T\mathbf{a}^{(j)} = \sum_{i=1}^{3} t'_{i,j}\mathbf{p}'^T\mathbf{a}^{(j'_i)} = \sum_{i=1}^{3} c_{j'_i} t'_{i,j},$$

and therefore

$$-q_j' = \sum_{i=1}^{3} c_{j'_i} t'_{i,j} - c_j.$$

The extended simplex tableau for the basis $\{1, 4, 3\}$ has now been computed, and the completed tableau is as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

The fact that $q'_j \geq 0$ for $j = 1, 2, 3, 4, 5$ shows that we have now found our basic optimal solution. Indeed the cost $\overline{C}$ of any feasible solution $\overline{\mathbf{x}}$ satisfies

$$\begin{aligned}
\overline{C} &= \mathbf{c}^T \overline{\mathbf{x}} = \mathbf{p}'^T A \overline{\mathbf{x}} + \mathbf{q}'^T \overline{\mathbf{x}} = \mathbf{p}'^T \mathbf{b} + \mathbf{q}'^T \overline{\mathbf{x}} \\
&= C' + \mathbf{q}'^T \overline{\mathbf{x}} \\
&= C' + \frac{76}{27} \overline{x}_2 + \frac{32}{27} \overline{x}_5,
\end{aligned}$$

where $\overline{x}_2 = (\overline{\mathbf{x}})_2$ and $\overline{x}_5 = (\overline{\mathbf{x}})_5$.

Therefore $\mathbf{x}'$ is a basic optimal solution to the linear programming problem, where

$$\mathbf{x}'^T = \begin{pmatrix} \frac{99}{27} & 0 & \frac{15}{27} & \frac{69}{27} & 0 \end{pmatrix}.$$

It is instructive to compare the pivot row and criterion row of the tableau for the basis $\{1, 2, 3\}$ with the corresponding rows of the tableau for the basis $\{1, 4, 3\}$.

These rows in the old tableau for the basis $\{1, 2, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

The corresponding rows in the new tableau for the basis $\{1, 4, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

If we examine the values of the criterion row in the new tableau we find that they are obtained from corresponding values in the criterion row of the old tableau by subtracting off the corresponding elements of the pivot row of the old tableau multiplied by the factor $\frac{76}{27}$. As a result, the new tableau has value 0 in the cell of the criterion row in column $\mathbf{a}^{(4)}$. Thus the same rule used to calculate values in other rows of the new tableau would also have yielded the correct elements in the criterion row of the tableau.

We now investigate the reasons why this is so.

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, 3, 4, 5$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j} \mathbf{a}^{(j_i)} = \sum_{i=1}^{3} t'_{i,j} \mathbf{a}^{(j'_i)},$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$. Moreover

$$t'_{2,j} = \frac{1}{t_{2,4}}\, t_{2,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, t_{2,j} \quad (i = 1, 3).$$

Now

$$\begin{aligned}
-q_j &= \sum_{i=1}^{3} c_{j_i} t_{i,j} - c_j \\
&= c_1 t_{1,j} + c_2 t_{2,j} + c_3 t_{3,j} - c_j, \\
-q_j' &= \sum_{i=1}^{3} c_{j_i'} t_{i,j}' - c_j. \\
&= c_1 t_{1,j}' + c_4 t_{2,j}' + c_3 t_{3,j}' - c_j.
\end{aligned}$$

Therefore

$$\begin{aligned} q_j - q_j' &= c_1(t_{1,j}' - t_{1,j}) + c_4 t_{2,j}' - c_2 t_{2,j} + c_3(t_{3,j}' - t_{3,j}) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) t_{2,j} \\ &= \frac{q_4}{t_{2,4}}\, t_{2,j} \end{aligned}$$

and thus

$$-q_j' = -q_j + \frac{q_4}{t_{2,4}}\, t_{2,j}$$

for $j = 1, 2, 3, 4, 5$.

Next we note that

$$\begin{aligned} C &= \sum_{i=1}^{3} c_{j_i} s_i = c_1 s_1 + c_2 s_2 + c_3 s_3, \\ C' &= \sum_{i=1}^{3} c_{j'_i} s'_i = c_1 s'_1 + c_4 s'_2 + c_3 s'_3. \end{aligned}$$

Therefore

$$\begin{aligned} C' - C &= c_1(s'_1 - s_1) + c_4 s'_2 - c_2 s_2 + c_3(s'_3 - s_3) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) s_2 \\ &= \frac{q_4}{t_{2,4}} s_2 \end{aligned}$$

and thus

$$C' = C + \frac{q_4}{t_{2,4}} s_2$$

for $k = 1, 2, 3$.

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$
\begin{aligned}
p_k &= \sum_{i=1}^{3} c_{j_i} r_{i,k} = c_1 r_{1,k} + c_2 r_{2,k} + c_3 r_{3,k}, \\
p'_k &= \sum_{i=1}^{3} c_{j'_i} r'_{i,k} = c_1 r'_{1,k} + c_4 r'_{2,k} + c_3 r'_{3,k}.
\end{aligned}
$$

Therefore

$$\begin{aligned} p'_k - p_k &= c_1(r'_{1,k} - r_{1,k}) + c_4 r'_{2,k} - c_2 r_{2,k} + c_3(r'_{3,k} - r_{3,k}) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) r_{2,k} \\ &= \frac{q_4}{t_{2,4}}\, r_{2,k} \end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_4}{t_{2,4}}\, r_{2,k}$$

for $k = 1, 2, 3$.

This completes the discussion of the structure and properties of the extended simplex tableau associated with the optimization problem under discussion.