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4.6. A Linear Tableau Example

Example

Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 13 \\ 13 \\ 20 \end{pmatrix},$$

$$\mathbf{c}^T = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix}.$$

As usual, we denote by $A_{i,j}$ the coefficient of the matrix A in the ith row and jth column, we denote by b_i the ith component of the m-dimensional vector $\mathbf{b}$, and we denote by c_j the jth component of the n-dimensional vector $\mathbf{c}$.

We let $\mathbf{a}^{(j)}$ be the m-dimensional vector specified by the jth column of the matrix A for $j = 1, 2, 3, 4, 5$. Then

$$\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix},$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(5)} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}.$$

A *basis* B for this linear programming problem is a subset of $\{1, 2, 3, 4, 5\}$ consisting of distinct integers j_1, j_2, j_3 for which the corresponding vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_3)}$ constitute a basis of the real vector space $\mathbb{R}^3$.

Given a basis B for the linear programming programming problem, where $B = \{j_1, j_2, j_3\}$, we denote by M_B the matrix whose columns are specified by the vectors $\mathbf{a}^{(j_1)}$, $\mathbf{a}^{(j_2)}$ and $\mathbf{a}^{(j_3)}$. Thus $(M_B)_{i,k} = A_{i,j_k}$ for $i = 1, 2, 3$ and $k = 1, 2, 3$. We also denote by $\mathbf{c}_B$ the 3-dimensional vector defined such that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & c_{j_3} \end{pmatrix}.$$

The ordering of the columns of M_B and $\mathbf{c}_B$ is determined by the ordering of the elements j_1, j_2 and j_3 of the basis. However we shall proceed on the basis that some ordering of the elements of a given basis has been chosen, and the matrix M_B and vector $\mathbf{c}_B$ will be determined so as to match the chosen ordering.

Let $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$, and let $B = \{j_1, j_2, j_3\} = \{1, 2, 3\}$. Then B is a basis of the linear programming problem, and the invertible matrix M_B determined by $\mathbf{a}^{(j_k)}$ for $k = 1, 2, 3$ is the following 3×3 matrix:—

$$M_B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{pmatrix}.$$

This matrix has determinant -23, and

$$M_B^{-1} = \frac{-1}{23} \begin{pmatrix} 13 & -4 & -7 \\ -6 & -7 & 5 \\ -8 & 6 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{13}{23} & \frac{4}{23} & \frac{7}{23} \\ \frac{6}{23} & \frac{7}{23} & -\frac{5}{23} \\ \frac{8}{23} & -\frac{6}{23} & \frac{1}{23} \end{pmatrix}.$$

Then

$$M_B^{-1}\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad M_B^{-1}\mathbf{a}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad M_B^{-1}\mathbf{a}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$M_B^{-1}\mathbf{a}^{(4)} = \begin{pmatrix} -\frac{24}{23} \\ \frac{27}{23} \\ \frac{13}{23} \end{pmatrix} \quad \text{and} \quad M_B^{-1}\mathbf{a}^{(5)} = \begin{pmatrix} -\frac{25}{23} \\ \frac{31}{23} \\ \frac{26}{23} \end{pmatrix}.$$

Also

$$M_B^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

It follows that $\mathbf{x}$ is a basic feasible solution of the linear programming problem, where

$$\mathbf{x}^T = \begin{pmatrix} 1 & 3 & 2 & 0 & 0 \end{pmatrix}.$$

The vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}$, $\mathbf{b}$, $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ can then be expressed as linear combinations of $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ with coefficients as recorded in the following tableau:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	.	.	.	.	.	.	.	.	.

There is an additional row at the bottom of the tableau. This row is the *criterion row* of the tableau. The values in this row have not yet been calculated, but, when calculated according to the rules described below, the values in the criterion row will establish whether the current basic feasible solution is optimal and, if not, how it can be improved.

Ignoring the criterion row, we can represent the structure of the remainder of the tableau in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\cdot$	$\cdot$	$\cdot$

We now employ the principles of the Simplex Method in order to determine whether or not the current basic feasible solution is optimal and, if not, how to improve it by changing the basis.

Let **p** be the 3-dimensional vector determined so that

$$\mathbf{p}^T = \mathbf{c}_B^T M_B^{-1}.$$

Then $\mathbf{p}^T M_B = \mathbf{c}_B^T$, and therefore $\mathbf{p}^T \mathbf{a}^{(j_k)} = c_{j_k}$ for $k = 1, 2, 3$. It follows that $(\mathbf{p}^T A)_j = c_j$ whenever $j \in B$. Putting in the relevant numerical values, we find that

$$\mathbf{p}^T M_B = \mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & c_{j_3} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix},$$

and therefore

$$\mathbf{p}^T = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix} M_B^{-1} = \begin{pmatrix} \frac{22}{23} & \frac{18}{23} & \frac{-3}{23} \end{pmatrix}.$$

We enter the values of p_1, p_2 and p_3 into the cells of the criterion row in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ respectively. The tableau with these values entered is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

The values in the criterion row in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ can be calculated from the components of the cost vector $\mathbf{c}$ and the values in these columns of the tableau. Indeed let $r_{i,k} = (M_B^{-1})_{i,k}$ for $i = 1, 2, 3$ and $k = 1, 2, 3$. Then each $r_{i,k}$ is equal to the value of the tableau element located in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{e}^{(k)}$. The definition of the vector $\mathbf{p}$ then ensures that

$$p_k = c_{j_1} r_{1,k} + c_{j_2} r_{2,k} + c_{j_3} r_{3,k}$$

for $k = 1, 2, 3$, where, for the current basis, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$.

The cost C of the current basic feasible solution $\mathbf{x}$ satisfies $C = \mathbf{c}^T\mathbf{x}$. Now $(\mathbf{p}^T A)_j = c_j$ for all $j \in B$, where $B = \{1, 2, 3\}$. Moreover the current basic feasible solution $\mathbf{x}$ satisfies $x_j = 0$ when $j \notin B$, where $x_j = (\mathbf{x})_j$ for $j = 1, 2, 3, 4, 5$. It follows that

$$
\begin{aligned}
C - \mathbf{p}^T\mathbf{b} &= \mathbf{c}^T\mathbf{x} - \mathbf{p}^T A\mathbf{x} = \sum_{j=1}^{5} (c_j - (\mathbf{p}^T A)_j) x_j \\
&= \sum_{j \in B} (c_j - (\mathbf{p}^T A)_j) x_j = 0,
\end{aligned}
$$

and thus

$$
C = \mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}.
$$

Putting in the numerical values, we find that $C = 20$.

We enter the cost C into the criterion row of the tableau in the column labelled by the vector $\mathbf{b}$. The resultant tableau is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	·	·	·	·	·	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

Let s_i denote the value recorded in the tableau in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{b}$ for $i = 1, 2, 3$. Then the construction of the tableau ensures that

$$\mathbf{b} = s_1\mathbf{a}^{(j_1)} + s_2\mathbf{a}^{(j_2)} + s_3\mathbf{a}^{(j_3)},$$

and thus $s_i = x_{j_i}$ for $i = 1, 2, 3$, where $(x_1, x_2, x_3, x_4, x_5)$ is the current basic feasible solution. It follows that

$$C = c_{j_1}s_1 + c_{j_2}s_2 + c_{j_3}s_3,$$

where, for the current basis, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$. Thus the cost of the current basic feasible solution can be calculated from the components of the cost vector $\mathbf{c}$ and the values recorded in the rows above the criterion row of the tableau in the column labelled by the vector $\mathbf{b}$.

We next determine a 5-dimensional vector $\mathbf{q}$ such that $\mathbf{c}^T = \mathbf{p}^T A + \mathbf{q}^T$. We find that

$$\begin{aligned}
-\mathbf{q}^T &= \mathbf{p}^T A - \mathbf{c}^T \\
&= \begin{pmatrix} \frac{22}{23} & \frac{18}{23} & \frac{-3}{23} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix} \\
&\qquad - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 4 & 3 & \frac{99}{23} & \frac{152}{23} \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & \frac{76}{23} & \frac{60}{23} \end{pmatrix}
\end{aligned}$$

Thus

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0, \quad q_4 = -\tfrac{76}{23}, \quad q_5 = -\tfrac{60}{23}.$$

The 4th and 5th components of the vector $\mathbf{q}$ are negative. It follows that the current basic feasible solution is not optimal. Indeed let $\overline{\mathbf{x}}$ be a basic feasible solution to the problem, and let $\overline{x}_j = (\overline{\mathbf{x}})_j$ for $j = 1, 2, 3, 4, 5$. Then the cost $\overline{C}$ of the feasible solution $\overline{\mathbf{x}}$ satisfies

$$\begin{aligned} \overline{C} &= \mathbf{c}^T \overline{\mathbf{x}} = \mathbf{p}^T A \overline{\mathbf{x}} + \mathbf{q}^T \overline{\mathbf{x}} = \mathbf{p}^T \mathbf{b} + \mathbf{q}^T \overline{\mathbf{x}} = C + \mathbf{q}^T \overline{\mathbf{x}} \\ &= C - \frac{76}{23} \overline{x}_4 - \frac{60}{23} \overline{x}_5. \end{aligned}$$

It follows that the basic feasible solution $\overline{\mathbf{x}}$ will have lower cost if either $\overline{x}_4 > 0$ or $\overline{x}_5 > 0$.

We enter the value of $-q_j$ into the criterion row of the tableau in the column labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, 3, 4, 5$. The completed tableau associated with basis $\{1, 2, 3\}$ is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

We refer to this tableau as the *extended simplex tableau* associated with the basis $\{1, 2, 3\}$.

The general structure of the extended simplex tableau is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	s_1	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$t_{2,3}$	$t_{2,4}$	$t_{2,5}$	s_2	$r_{2,1}$	$r_{2,2}$	$r_{2,3}$
$\mathbf{a}^{(j_3)}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$	$t_{3,4}$	$t_{3,5}$	s_3	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$
	$-q_1$	$-q_2$	$-q_3$	$-q_4$	$-q_5$	C	p_1	p_2	p_3

where j_1, j_2 and j_3 are the elements of the current basis, and where the coefficients $t_{i,j}$ s_i and $r_{i,k}$ are determined so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)}, \quad \mathbf{b} = \sum_{i=1}^{3} s_i\mathbf{a}^{(j_i)}, \quad \mathbf{e}^{(k)} = \sum_{i=1}^{3} r_{i,k}\mathbf{a}^{(j_i)}$$

for $j = 1, 2, 3, 4, 5$ and $k = 1, 2, 3$.

The coefficients of the criterion row can then be calculated according to the following formulae:—

$$p_k = \sum_{i=1}^{3} c_{j_i} r_{i,k}, \quad C = \sum_{i=1}^{3} p_i b_i, \quad -q_j = \sum_{i=1}^{3} p_i A_{i,j} - c_j.$$

The extended simplex tableau can then be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

The values in the criterion row in any column labelled by some $\mathbf{a}^{(j)}$ can also be calculated from the values in the relevant column in the rows above the criterion row.

To see this we note that the value entered into the tableau in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{a}^{(j)}$ is equal to $t_{i,j}$, where $t_{i,j}$ is the coefficient in the ith row and jth column of the matrix $M_B^{-1}A$. Also $\mathbf{p}^T = \mathbf{c}_B^T M_B^{-1}$, where $(\mathbf{c}_B)_i = c_{j_i}$ for $i = 1, 2, 3$. It follows that

$$\mathbf{p}^T A = \mathbf{c}_B^T M_B^{-1} A = \sum_{i=1}^{3} c_{j_i} t_{i,j}.$$

Therefore

$$\begin{aligned} -q_j &= (\mathbf{p}^T A)_j - c_j \\ &= c_{j_1} t_{1,j} + c_{j_2} t_{2,j} + c_{j_3} t_{3,j} - c_j \end{aligned}$$

for $j = 1, 2, 3, 4, 5$.

The coefficients of the criterion row can then be calculated according to the formulae

$$p_k = \sum_{i=1}^{3} c_{j_i} r_{i,k}, \quad C = \sum_{i=1}^{3} c_{j_i} s_i, \quad -q_j = \sum_{i=1}^{3} c_{j_i} t_{i,j} - c_j.$$

The extended simplex tableau can therefore also be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$