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4. The Simplex Method

4.1. Vector Inequalities and Notational Conventions

Let $\mathbf{v}$ be an element of the real vector space $\mathbb{R}^n$. We denote by $(\mathbf{v})_j$ the jth component of the vector $\mathbf{v}$. The vector $\mathbf{v}$ can be represented in the usual fashion as an n-tuple $(v_1, v_2, \ldots, v_n)$, where $v_j = (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$. However where an n-dimensional vector appears in matrix equations it will usually be considered to be an $n \times 1$ column vector. The row vector corresponding to an element $\mathbf{v}$ of $\mathbb{R}^n$ will be denoted by $\mathbf{v}^T$ because, considered as a matrix, it is the transpose of the column vector representing $\mathbf{v}$. We denote the zero vector (in the appropriate dimension) by $\mathbf{0}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors belonging to the real vector space $\mathbb{R}^n$ for some positive integer n. We write $\mathbf{x} \leq \mathbf{y}$ (and $\mathbf{y} \geq \mathbf{x}$) when $(\mathbf{x})_j \leq (\mathbf{y})_j$ for $j = 1, 2, \ldots, n$. Also we write $\mathbf{x} \ll \mathbf{y}$ (and $\mathbf{y} \gg \mathbf{x}$) when $(\mathbf{x})_j < (\mathbf{y})_j$ for $j = 1, 2, \ldots, n$.

These notational conventions ensure that $\mathbf{x} \geq \mathbf{0}$ if and only if $(\mathbf{x})_j \geq 0$ for $j = 1, 2, \ldots, n$.

The scalar product of two n-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ can be represented as the matrix product $\mathbf{u}^T\mathbf{v}$. Thus

$$\mathbf{u}^T\mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $u_j = (\mathbf{u})_j$ and $v_j = (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$.

Given an $m \times n$ matrix A, where m and n are positive integers, we denote by $(A)_{i,j}$ the coefficient in the ith row and jth column of the matrix A.

4.2. Feasible and Optimal Solutions

A general linear programming problem is one that seeks values of real variables $x_1, x_2, \ldots, x_n$ that maximize or minimize some *objective function*

$$c_1 x_1 + c_2 x_2 + \cdots c_n x_n$$

that is a linear functional of $x_1, x_2, \ldots, x_n$ determined by real constants $c_1, c_2, \ldots, c_n$, where the variables $x_1, x_2, \ldots, x_n$ are subject to a finite number of *constraints* that each place bounds on the value of some linear functional of the variables.

These constraints can then be numbered from 1 to m, for an appropriate value of m, such that, for each value of i between 1 and m, the ith constraint takes the form of an equation or inequality that can be expressed in one of the following three forms:—

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = b_i,$$

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \geq b_i,$$

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \leq b_i$$

for appropriate values of the real constants $a_{i,1}, a_{i,2}, \ldots, a_{i,n}$ and b_i. In addition some, but not necessarily all, of the variables $x_1, x_2, \ldots, x_n$ may be required to be non-negative. (Of course a constraint requiring a variable to be non-negative can be expressed by an inequality that conforms to one of the three forms described above. Nevertheless constraints that simply require some of the variables to be non-negative are usually listed separately from the other constraints.)

Definition

Consider a general linear programming problem with n real variables $x_1, x_2, \ldots, x_n$ whose objective is to maximize or minimize some objective function subject to appropriate constraints. A *feasible solution* of this linear programming problem is specified by an n-dimensional vector $\mathbf{x}$ whose components satisfy the constraints but do not necessarily maximize or minimize the objective function.

Definition

Consider a general linear programming problem with n real variables $x_1, x_2, \ldots, x_n$ whose objective is to maximize or minimize some objective function subject to appropriate constraints. A *optimal solution* of this linear programming problem is specified by an n-dimensional vector $\mathbf{x}$ that is a feasible solution that optimizes the value of the objective function amongst all feasible solutions to the linear programming problem.

4.3. Programming Problems in Dantzig Standard Form

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, and let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ be vectors of dimensions m and n respectively. We consider the following linear programming problem:—

> *Determine an n-dimensional vector* $\mathbf{x}$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to the constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

We refer to linear programming problems presented in this form as being in *Dantzig standard form*. We refer to the $m \times n$ matrix A, the m-dimensional vector $\mathbf{b}$ and the n-dimensional vector $\mathbf{c}$ as the *constraint matrix*, *target vector* and *cost vector* for the linear programming problem.

Remark

Nomenclature in Linear Programming textbooks varies. Problems presented in the above form are those to which the basic algorithms of George B. Dantzig's *Simplex Method* are applicable. In the series of textbooks by George B. Dantzig and Mukund N. Thapa entitled *Linear Programming*, such problems are said to be in *standard form*. In the textbook *Introduction to Linear Programming* by Richard B. Darst, such problems are said to be *standard-form LP*. On the other hand, in the textbook *Methods of Mathematical Economics* by Joel N. Franklin, such problems are said to be in *canonical form*, and the term *standard form* is used for problems which match the form above, except that the vector equality $A\mathbf{x} = \mathbf{b}$ is replaced by a vector inequality $A\mathbf{x} \geq \mathbf{b}$.

Accordingly the term *Danztig standard form* is used in these notes both to indicate that such problems are in *standard form* at that term is used by textbooks of which Dantzig is the author, and also to emphasize the connection with the contribution of Dantzig in creating and popularizing the *Simplex Method* for the solution of linear programming problems.

A linear programming problem in Dantzig standard form specified by an $m \times n$ constraint matrix A of rank m, an m-dimensional target vector $\mathbf{b}$ and an n-dimensional cost vector $\mathbf{c}$ has the objective of finding values of real variables $x_1, x_2, \ldots, x_n$ that minimize the value of the *cost*

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subject to constraints

$$\begin{array}{rcl} A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n & = & b_1, \\ A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n & = & b_2, \\ & \vdots & \\ A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n & = & b_m \end{array}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \ldots, \quad x_n \geq 0.$$

In the above programming problem, the function sending the n-dimensional vector $\mathbf{x}$ to the corresponding *cost* $\mathbf{c}^T\mathbf{x}$ is the objective function for the problem. A feasible solution to the problem consists of an n-dimensional vector $(x_1, x_2, \ldots, x_n)$ whose components satisfy the above constraints but do not necessarily minimize cost. An optimal solution is a feasible solution whose cost does not exceed that of any other feasible solution.

4.4. Basic Feasible Solutions

We define the notion of a *basis* for a linear programming problem in Dantzig standard form.

Definition

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:

> *find* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

For each integer j between 1 and n, let $\mathbf{a}^{(j)}$ denote the m-dimensional vector determined by the jth column of the matrix A, so that $(\mathbf{a}^{(j)})_i = (A)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. A *basis* for this linear programming problem is a set consisting of m distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n for which the corresponding vectors

$$\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$$

constitute a basis of the vector space $\mathbb{R}^m$.

We next define what is meant by saying that a feasible solution of a programming problem Dantzig standard form is a *basic feasible solution* for the programming problem.

Definition

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:—

> *find* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

A feasible solution $\mathbf{x}$ for this programming problem is said to be *basic* if there exists a basis B for the linear programming problem such that $(\mathbf{x})_j = 0$ when $j \notin B$.

Lemma 4.1

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:

> *find $\mathbf{x} \in \mathbb{R}^n$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

Let $\mathbf{a}^{(j)}$ denote the vector specified by the jth column of the matrix A for $j = 1, 2, \ldots, n$. Let $\mathbf{x}$ be a feasible solution of the linear programming problem. Suppose that the m-dimensional vectors $\mathbf{a}^{(j)}$ for which $(\mathbf{x})_j > 0$ are linearly independent. Then $\mathbf{x}$ is a basic feasible solution of the linear programming problem.

Proof

Let $\mathbf{x}$ be a feasible solution to the programming problem, let $x_j = (\mathbf{x})_j$ for all $j \in J$, where $J = \{1, 2, \ldots, n\}$, and let $K = \{j \in J : x_j > 0\}$. If the vectors $\mathbf{a}^{(j)}$ for which $j \in K$ are linearly independent then basic linear algebra ensures that further vectors $\mathbf{a}^{(j)}$ can be added to the linearly independent set $\{\mathbf{a}^{(j)} : j \in K\}$ so as to obtain a finite subset of $\mathbb{R}^m$ whose elements constitute a basis of that vector space (see Proposition 2.2). Thus exists a subset B of J satisfying $K \subset B \subset J$ such that the m-dimensional vectors $\mathbf{a}^{(j)}$ for which $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$. Moreover $(\mathbf{x})_j = 0$ for all $j \in J \setminus B$. It follows that $\mathbf{x}$ is a basic feasible solution to the linear programming problem, as required. ∎

Theorem 4.2

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:

> *find $\mathbf{x} \in \mathbb{R}^n$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

If there exists a feasible solution to this programming problem then there exists a basic feasible solution to the problem. Moreover if there exists an optimal solution to the programming problem then there exists a basic optimal solution to the problem.

Proof

Let $J = \{1, 2, \ldots, n\}$, and let $\mathbf{a}^{(j)}$ denote the vector specified by the jth column of the matrix A for all $j \in J$.

Let $\mathbf{x}$ be a feasible solution to the programming problem, let $x_j = (\mathbf{x})_j$ for all $j \in J$, and let $K = \{j \in J : x_j > 0\}$. Suppose that $\mathbf{x}$ is not basic. Then the vectors $\mathbf{a}^{(j)}$ for which $j \in K$ must be linearly dependent. We show that there then exists a feasible solution with fewer non-zero components than the given feasible solution $\mathbf{x}$.

Now there exist real numbers y_j for $j \in K$, not all zero, such that $\sum_{j \in K} y_j \mathbf{a}^{(j)} = \mathbf{0}$, because the vectors $\mathbf{a}^{(j)}$ for $j \in K$ are linearly dependent. Let $y_j = 0$ for all $j \in J \setminus K$, and let $\mathbf{y} \in \mathbb{R}^n$ be the n-dimensional vector satisfying $(\mathbf{y})_j = y_j$ for $j = 1, 2, \ldots, n$. Then

$$A\mathbf{y} = \sum_{j \in J} y_j \mathbf{a}^{(j)} = \sum_{j \in K} y_j \mathbf{a}^{(j)} = \mathbf{0}.$$

It follows that $A(\mathbf{x} - \lambda \mathbf{y}) = \mathbf{b}$ for all real numbers λ, and thus $\mathbf{x} - \lambda \mathbf{y}$ is a feasible solution to the programming problem for all real numbers λ for which $\mathbf{x} - \lambda \mathbf{y} \geq \mathbf{0}$.

Now $\mathbf{y}$ is a non-zero vector. Replacing $\mathbf{y}$ by $-\mathbf{y}$, if necessary, we can assume, without loss of generality, that at least one component of the vector $\mathbf{y}$ is positive. Let

$$\lambda_0 = \text{minimum}\left(\frac{x_j}{y_j} : j \in K \text{ and } y_j > 0\right),$$

and let j_0 be an element of K for which $\lambda_0 = x_{j_0}/y_{j_0}$. Then $\dfrac{x_j}{y_j} \geq \lambda_0$ for all $j \in J$ for which $y_j > 0$. Multiplying by the positive number y_j, we find that $x_j \geq \lambda_0 y_j$ and thus $x_j - \lambda_0 y_j \geq 0$ when $y_j > 0$. Also $\lambda_0 > 0$ and $x_j \geq 0$, and therefore $x_j - \lambda_0 y_j \geq 0$ when $y_j \leq 0$. Thus $x_j - \lambda_0 y_j \geq 0$ for all $j \in J$. Also $x_{j_0} - \lambda_0 y_{j_0} = 0$, and $x_j - \lambda_0 y_j = 0$ for all $j \in J \setminus K$. Let $\mathbf{x}' = \mathbf{x} - \lambda_0 \mathbf{y}$. Then $\mathbf{x}' \geq \mathbf{0}$ and $A\mathbf{x}' = \mathbf{b}$, and thus $\mathbf{x}'$ is a feasible solution to the linear programming problem with fewer non-zero components than the given feasible solution.

Suppose in particular that the feasible solution $\mathbf{x}$ is optimal. Now there exist both positive and negative values of λ for which $\mathbf{x} - \lambda \mathbf{y} \geq \mathbf{0}$. If it were the case that $\mathbf{c}^T \mathbf{y} \neq 0$ then there would exist values of λ for which both $\mathbf{x} - \lambda \mathbf{y} \geq \mathbf{0}$ and $\lambda \mathbf{c}^T \mathbf{y} > 0$. But then $\mathbf{c}^T (x - \lambda \mathbf{y}) < \mathbf{c}^T \mathbf{x}$, contradicting the optimality of $\mathbf{x}$. It follows that $\mathbf{c}^T \mathbf{y} = 0$, and therefore $\mathbf{x} - \lambda \mathbf{y}$ is an optimal solution of the linear programming problem for all values of λ for which $\mathbf{x} - \lambda \mathbf{y} \geq \mathbf{0}$. The previous argument then shows that there exists a real number λ_0 for which $\mathbf{x} - \lambda_0 \mathbf{y}$ is an optimal solution with fewer non-zero components than the given optimal solution $\mathbf{x}$.

We have shown that if there exists a feasible solution $\mathbf{x}$ which is not basic then there exists a feasible solution with fewer non-zero components than $\mathbf{x}$. It follows that if a feasible solution $\mathbf{x}$ is chosen such that it has the smallest possible number of non-zero components then it is a basic feasible solution of the linear programming problem.

Similarly we have shown that if there exists an optimal solution $\mathbf{x}$ which is not basic then there exists an optimal solution with fewer non-zero components than $\mathbf{x}$. It follows that if an optimal solution $\mathbf{x}$ is chosen such that it has the smallest possible number of non-zero components then it is a basic optimal solution of the linear programming problem. $\blacksquare$