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David R. Wilkins

3.11. Formal Analysis of the Solution of the Transportation Problem

We now describe in general terms the method for solving a transportation problem in which total supply equals total demand.

We suppose that an initial basic feasible solution has been obtained. We apply an iterative method (based on the general Simplex Method for the solution of linear programming problems) that will test a basic feasible solution for optimality and, in the event that the feasible solution is shown not to be optimal, establishes information that (with the exception of certain 'degenerate' cases of the transportation problem) enables one to find a basic feasible solution with lower cost. Iterating this procedure a finite number of times, one should arrive at a basic feasible solution that is optimal for the given transportation problem.

We suppose that the given instance of the Transportation Problem involves m suppliers and n recipients. The required supplies are specified by non-negative real numbers $s_1, s_2, \ldots, s_m$, and the required demands are specified by non-negative real numbers $d_1, d_2, \ldots, d_n$. We further suppose that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. A *feasible solution* is represented by non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. A subset B of $I \times J$ is a *basis* for the transportation problem if and only if, given any real numbers $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_n$, where $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exist uniquely determined real numbers $\overline{x}_{i,j}$ for $i \in I$ and $j \in J$ such that $\sum_{j=1}^{n} \overline{x}_{i,j} = y_i$ for $i \in I$, $\sum_{i=1}^{m} \overline{x}_{i,j} = z_j$ for $j \in J$, where $\overline{x}_{i,j} = 0$ whenever $(i,j) \notin B$.

A feasible solution $(x_{i,j})$ is said to be a basic feasible solution associated with the basis B if and only if $x_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i,j) \notin B$.

Let $x_{i,j}$ be a non-negative real number for each $i \in I$ and $j \in J$. Suppose that $(x_{i,j})$ is a basic feasible solution to the transportation problem associated with basis B, where $B \subset I \times J$.

The cost associated with a feasible solution $(x_{i,j}$ is given by $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}$, where the constants $c_{i,j}$ are real numbers for all $i \in I$ and $j \in J$. A feasible solution for a transportation problem is an optimal solution if and only if it minimizes cost amongst all feasible solutions to the problem.

In order to test for optimality of a basic feasible solution $(x_{i,j})$ associated with a basis B, we determine real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the property that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. (Proposition 3.10 below guarantees that, given any basis B, it is always possible to find the required quantities u_i and v_j.) Having calculated these quantities u_i and v_j we determine the values of $q_{i,j}$, where $q_{i,j} = c_{i,j} - v_j + u_i$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $(i,j) \in B$.

We claim that a basic feasible solution $(x_{i,j})$ associated with the basis B is optimal if and only if $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$. This is a consequence of the identity established in the following proposition.

Proposition 3.8

Let $x_{i,j}$, $c_{i,j}$ and $q_{i,j}$ be real numbers defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers. Suppose that

$$c_{i,j} = v_j - u_i + q_{i,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j},$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ for $i = 1, 2, \ldots, m$ and $d_j = \sum_{i=1}^{m} x_{i,j}$ for $j = 1, 2, \ldots, n$.

Proof

The definitions of the relevant quantities ensure that

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} &= \sum_{i=1}^{m}\sum_{j=1}^{n}(v_j - u_i + q_{i,j})x_{i,j} \\
&= \sum_{j=1}^{n}\left(v_j \sum_{i=1}^{m} x_{i,j}\right) - \sum_{i=1}^{m}\left(u_i \sum_{j=1}^{n} x_{i,j}\right) \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} \\
&= \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j},
\end{aligned}$$

as required. ∎

Corollary 3.9

Let m and n be integers, and let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. Let $x_{i,j}$ and $c_{i,j}$ be real numbers defined for all $i \in I$ and $j \in I$, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers. Suppose that $c_{i,j} = v_j - u_i$ for all $(i, j) \in I \times J$ for which $x_{i,j} \neq 0$. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ for $i = 1, 2, \ldots, m$ and $d_j = \sum_{i=1}^{m} x_{i,j}$ for $j = 1, 2, \ldots, n$.

Proof

Let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $x_{i,j} \neq 0$. It follows from this that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} = 0.$$

It then follows from Proposition 3.8 that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j - u_i + q_{i,j}) x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

as required. ∎

Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let the subset B of $I \times J$ be a basis for a transportation problem with m suppliers and n recipients. Let the cost of a feasible solution $(\overline{x}_{i,j})$ be $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j}$. Now $\sum_{j=1}^{n} \overline{x}_{i,j} = s_i$ and $\sum_{i=1}^{m} \overline{x}_{i,j} = d_j$, where the quantities s_i and d_j are determined by the specification of the problem and are the same for all feasible solutions of the problem. Let quantities u_i for $i \in I$ and v_j for $j \in J$ be determined such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, and let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ for all $(i,j) \in B$.

It follows from Proposition 3.8 that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

Now if the quantities $x_{i,j}$ for $i \in I$ and $j \in J$ constitute a basic feasible solution associated with the basis B then $x_{i,j} = 0$ whenever $(i,j) \notin B$. It follows that $\sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} = 0$, and therefore

$$\sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i = C,$$

where

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

The cost $\overline{C}$ of the feasible solution $(\overline{x}_{i,j})$ then satisfies the equation

$$\overline{C} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} \overline{x}_{i,j} = C + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} \overline{x}_{i,j}.$$

If $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$, then the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ is bounded below by the cost of the basic feasible solution $(x_{i,j})$. It follows that, in this case, the basic feasible solution $(x_{i,j})$ is optimal.

Suppose that (i_0, j_0) is an element of $I \times J$ for which $q_{i_0,j_0} < 0$. Then $(i_0, j_0) \notin B$. There is no basis for the transportation problem that includes the set $B \cup \{(i_0, j_0)\}$. A straightforward application of Proposition 3.6 establishes the existence of quantities $y_{i,j}$ for $i \in I$ and $j \in J$ such that $y_{i_0,j_0} = 1$ and $y_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B \cup \{(i_0, j_0)\}$.

Let the $m \times n$ matrices X and Y be defined so that $(X)_{i,j} = x_{i,j}$ and $(Y)_{i,j} = y_{i,j}$ for all $i \in I$ and $j \in J$. Suppose that $x_{i,j} > 0$ for all $(i,j) \in B$. Then the components of X in the basis positions are strictly positive. It follows that, if λ is positive but sufficiently small, then the components of the matrix $X + \lambda Y$ in the basis positions are also strictly positive, and therefore the components of the matrix $X + \lambda Y$ are non-negative for all sufficiently small non-negative values of λ. There will then exist a maximum value λ_0 that is an upper bound on the values of λ for which all components of the matrix $X + \lambda Y$ are non-negative. It is then a straightforward exercise in linear algebra to verify that $X + \lambda_0 Y$ is another basic feasible solution associated with a basis that includes (i_0, j_0) together with all but one of the elements of the basis B.

Moreover the cost of this new basic feasible solution is $C + \lambda_0 q_{i_0,j_0}$, where C is the cost of the basic feasible solution represented by the matrix X. Thus if $q_{i_0,j_0} < 0$ then the cost of the new basic feasible solution is lower than that of the basic feasible solution X from which it was derived.

Suppose that, for all basic feasible solutions of the given Transportation problem, the coefficients of the matrix specifying the basic feasible solution are strictly positive at the basis positions. Then a finite number of iterations of the procedure discussed above with result in an basic optimal solution of the given transportation problem. Such problems are said to be *non-degenerate*.

However if it turns out that a basic feasible solution $(x_{i,j})$ associated with a basis B satisfies $x_{i,j} = 0$ for some $(i,j) \in B$, then we are in a *degenerate* case of the transportation problem. The theory of degenerate cases of linear programming problems is discussed in detail in textbooks that discuss the details of linear programming algorithms.

We now establish the proposition that guarantees that, given any basis B, there exist quantities $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ such that the costs $c_{i,j}$ associated with the given transportation problem satisfy $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. This result is an essential component of the method described here for testing basic feasible solutions to determine whether or not they are optimal.

Proposition 3.10

Let m and n be integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let B be a subset of $I \times J$ that is a basis for the transportation problem with m suppliers and n recipients. For each $(i, j) \in B$ let $c_{i,j}$ be a corresponding real number. Then there exist real numbers u_i for $i \in I$ and v_j for $j \in J$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. Moreover if $\overline{u}_i$ and $\overline{v}_j$ are real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i, j) \in B$, then there exists some real number k such that $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$.

Proof

Let

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in B\}.$$

It follows from the definition of bases for transportation problems that there exist unique $m \times n$ matrices $S_1, S_2, \ldots, S_m$ belonging to M_B, where S_1 is the zero matrix, and where, for each integer i satisfying $1 < i \leq m$, the matrix S_k has the properties that

$$\sum_{\ell=1}^{n} (S_i)_{k,\ell} = \begin{cases} 1 & \text{if } k = 1, \\ -1 & \text{if } k = i, \\ 0 & \text{if } k \in I \setminus \{1, i\}, \end{cases}$$

and

$$\sum_{k=1}^{m} (S_i)_{k,\ell} = 0 \text{ for all } \ell \in J.$$

Also there exist unique $m \times n$ matrices $T_1, T_2, \ldots, T_m$ belonging to M_B where, for each integer j satisfying $1 \leq j \leq n$, the matrix T_j has the properties that

$$\sum_{j=1}^{n} (T_j)_{k,l} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \in I \setminus \{1\}, \end{cases}$$

and

$$\sum_{i=1}^{m} (T_j)_{k,\ell} = \begin{cases} 1 & \text{if } \ell = j, \\ 0 & \text{if } \ell \in J \setminus \{j\}, \end{cases}$$

Let

$$u_i = \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{k,\ell} (S_i)_{k,\ell}$$

for $i = 1, 2, \ldots, m$ and

$$v_j = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (T_j)_{k,\ell}.$$

for $j = 1, 2, \ldots, n$. We claim the that numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ have the required properties.

Let X be an $m \times n$ matrix belonging to M_B, and let

$$y_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for all } i \in I$$

and

$$z_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for all } j \in J,$$

and let

$$\overline{X} = \sum_{\ell=1}^{n} z_\ell T_\ell - \sum_{k=1}^{m} y_k S_k.$$

Then

$$\sum_{i=1}^{m} (\overline{X})_{i,j} = z_j \quad \text{for all } j \in J.$$

and

$$\sum_{j=1}^{n} (\overline{X})_{i,j} = y_i \quad \text{for all } i \in I \setminus \{1\},$$

Moreover

$$\sum_{j=1}^{n} (\overline{X})_{1,j} = \sum_{\ell=1}^{n} z_\ell - \sum_{k=2}^{m} y_k = y_1,$$

because $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$.

But the definition of bases for transportation problems ensures that X is the unique $m \times n$ matrix belonging to M_B with the properties that $\sum_{j=1}^{n} (X)_{i,j} = y_i$ for all $i \in I$ and $\sum_{i=1}^{m} (X)_{i,j} = z_j$ for all $j \in J$. It follows that

$$X = \overline{X} = \sum_{j=1}^{n} z_j T_j - \sum_{i=1}^{m} y_i S_i,$$

and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (X)_{k,\ell} = \sum_{j=1}^{n} z_j v_j - \sum_{i=1}^{m} y_i u_i.$$

Let $(i,j) \in B$. Then $E^{(i,j)} \in M_B$, where

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

It follows from the result just obtained that

$$c_{i,j} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (E^{(i,j)})_{k,\ell} = v_j - u_i.$$

We have thus shown that, given any basis B for the transportation problem with m suppliers and n recipients, there exist real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the required property that

$$c_{i,j} = v_j - u_i \quad \text{for all } (i,j) \in B..$$

3. The Transportation Problem (continued)

Now let $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m$ and $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n$ be real numbers with the property that

$$c_{i,j} = \overline{v}_j - \overline{u}_i \quad \text{for all } (i,j) \in B..$$

Then $b_j - a_i = 0$ for all $(i,j) \in B$, where $a_i = \overline{u}_i - u_i$ for $i = 1, 2, \ldots, m$ and $b_j = \overline{v}_j - v_j$ for $j = 1, 2, \ldots, n$, and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(E^{i,j})_{k,\ell} = 0$$

for all $(i,j) \in B$. Now the $m \times n$ matrices $E^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the vector space M_B. It follows that

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(X)_{k,\ell} = 0$$

for all $X \in M_B$.

In particular

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n}(b_\ell - a_k)(S_i)_{k,\ell} = 0$$

for $i = 2, 3, \ldots, m$, and

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n}(b_\ell - a_k)(T_j)_{k,\ell} = 0$$

for $j = 1, 2, \ldots, n$.

But it follows from the definitions of the matrices $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_n$ that

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{\ell=1}^{n} b_\ell (S_i)_{k,\ell} &= \sum_{\ell=1}^{n} \left(b_\ell \sum_{k=1}^{m} (S_i)_{k,\ell} \right) = 0, \\
\sum_{k=1}^{m} \sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m} \left(a_k \sum_{\ell=1}^{n} (S_i)_{k,\ell} \right) = a_1 - a_i
\end{aligned}$$

for $i = 2, 3, \ldots, m$, and

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{\ell=1}^{n} b_\ell (T_j)_{k,\ell} &= \sum_{\ell=1}^{n} \left(b_\ell \sum_{k=1}^{m} (T_j)_{k,\ell} \right) = b_j, \\
\sum_{k=1}^{m} \sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m} \left(a_k \sum_{\ell=1}^{n} (S_i)_{k,\ell} \right) = a_1
\end{aligned}$$

for $j = 1, 2, \ldots, n$.

It follows that $a_i - a_1 = 0$ for $i = 2, \ldots, n$ and $b_j - a_1 = 0$ for $j = 1, 2, \ldots, n$. Thus if $k = a_1$ then $\overline{u}_i = u_i + a_i = u_i + k$ for $i = 1, 2, \ldots, m$ and $\overline{v}_j = v_j + b_j = v_j + k$ for $j = 1, 2, \ldots, n$, as required. $\blacksquare$