MA3484—Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 9 (February 3, 2017)

David R. Wilkins

3.10. A Method for finding Basic Optimal Solutions

We continue with the study of the optimization problem introduced in the discussion of the minimum cost method.

Example

We seek to determine non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

subject to the constraints

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad (i = 1, 2, 3, 4)$$

and

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad (j = 1, 2, 3),$$

where

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We have found an initial basic feasible solution by the Minimum Cost Method. This solution satisfies $x_{i,j} = (X)_{i,j}$ for all i and j, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

We next determine whether this initial basic feasible solution is an optimal solution, and, if not, how to adjust the basis to obtain a solution of lower cost.

We determine u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, where B is the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	•	4	•	16		0
		0		0		?	
2	3		7		2	•	?
		?		?		0	
3	13	•	8		6	•	?
		0		?		0	
4	5	•	7		8		?
		0		?		?	
v_j	?		?		?		

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$. After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 ?	7 ?	2 • 0	?
3	13 • 0	8 ?	6 • 0	?
4	5 • 0	7 ?	8 ?	?
v_j	8	4	?	

Then $v_1 = 8$, $(3,1) \in B$ and $(4,1) \in B$ force $u_3 = -5$ and $u_4 = 3$. After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	•	4	•	16		0
		0		0		?	
2	3		7		2	•	?
		?		?		0	
3	13	•	8		6	•	-5
		0		?		0	
4	5	•	7		8		3
		0		?		?	
v_j	8		4		?		

Then $u_3 = -5$ and $(3,3) \in B$ force $v_3 = 1$. After entering this value the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 ?	7 ?	2 • 0	?
3	13 • 0	8 ?	6 • 0	-5
4	5 • 0	7 ?	8 ?	3
v_j	8	4	1	

Then $v_3 = 1$ and $(2,3) \in B$ force $u_2 = -1$.

After entering the numbers u_i and v_j, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 ?	7 ?	2 • 0	-1
3	13 • 0	8 ?	6 • 0	-5
4	5 • 0	7 ?	8 ?	3
v_j	8	4	1	

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 15$, $q_{2,1} = -6$, $q_{2,2} = 2$, $q_{3,2} = -1$, $q_{4,2} = 6$ and $q_{4,3} = 10$.

3. The Transportation Problem (continued)

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 15	0
2	3 -6	7 2	2 • 0	-1
3	13 • 0	8 -1	6 • 0	-5
4	5 • 0	7 6	8 10	3
v_j	8	4	1	

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. To see this, suppose that the numbers $\overline{x}_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ constitute a feasible solution to the given problem. Then $\sum_{j=1}^{3} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4$. It follows that

$$\begin{aligned} \sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} \overline{x}_{i,j} &= \sum_{i=1}^{4} \sum_{j=1}^{3} (v_j - u_i + q_{i,j}) \overline{x}_{i,j} \\ &= \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4} \sum_{j=1}^{3} q_{i,j} \overline{x}_{i,j}. \end{aligned}$$

Applying this identity to the initial basic feasible solution, we find that $\sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i = 238$, given that 238 is the cost of the initial basic feasible solution. Thus the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 238 + 15\overline{x}_{1,3} - 6\overline{x}_{2,1} + 2\overline{x}_{2,2} - \overline{x}_{3,2} + 6\overline{x}_{4,2} + 10\overline{x}_{4,3}.$$

One could construct feasible solutions with $\overline{x}_{2,1} < 0$ and $\overline{x}_{i,j} = 0$ for $(i,j) \notin B \cup \{(2,1)\}$, and the cost of such feasible solutions would be lower than that of the initial basic solution. We therefore seek to bring $(2,1)$ into the basis, removing some other element of the basis to ensure that the new basis corresponds to a feasible basic solution.

The procedure for achieving this requires us to determine a 4×3 matrix Y satisfying the following conditions:—

- $y_{2,1} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(2, 1)\}$;
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	
1	? •	? •		0
2	1 ◦		? •	0
3	? •		? •	0
4	? •			0
	0	0	0	0

The constraints that $y_{2,1} = 1$, $y_{i,j} = 0$ when $(i,j) \notin B$ and the constraints requiring the rows and columns to sum to zero determine the values of $y_{i,j}$ for all $y_{i,j} \in B$. These values are recorded in the following tableau:—

$y_{i,j}$	1		2		3		
1	0	$\bullet$	0	$\bullet$			0
2	1	$\circ$			-1	$\bullet$	0
3	-1	$\bullet$			1	$\bullet$	0
4	0	$\bullet$					0
	0		0		0		0

We now determine those values of λ for which $X + \lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 1 & 12 & 0 \\ \lambda & 0 & 8 - \lambda \\ 5 - \lambda & 0 & 6 + \lambda \\ 13 & 0 & 0 \end{pmatrix}.$$

In order to drive down the cost as far as possible, we should make λ as large as possible, subject to the requirement that all the coefficients of the above matrix should be non-negative numbers.

Accordingly we take $\lambda = 5$. Our new basic feasible solution X is then as follows:—

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

We regard X as the current feasible basic solution.

The cost of the current feasible basic solution X is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 3 \times 5 + 2 \times 3 + 6 \times 11 \\ & \qquad + 5 \times 13 \\ = & \quad 8 + 48 + 15 + 6 + 66 + 65 \\ = & \quad 208. \end{aligned}$$

The cost has gone down by 30, as one would expect (the reduction in the cost being $-\lambda q_{2,1}$ where $\lambda = 5$ and $q_{2,1} = -6$).

The current basic feasible solution X is associated with the basis B where

$$B = \{(1,1),\ (1,2),\ (2,1),\ (2,3),\ (3,3),\ (4,1)\}.$$

We now determine, for the current basis B values u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

3. The Transportation Problem (continued)

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 $\bullet$ 0	4 $\bullet$ 0	16 ?	0
2	3 $\bullet$ 0	7 ?	2 $\bullet$ 0	?
3	13 ?	8 ?	6 $\bullet$ 0	?
4	5 $\bullet$ 0	7 ?	8 ?	?
v_j	?	?	?	

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$.

Then $v_1 = 8$, $(2,1) \in B$ and $(4,1) \in B$ force $u_2 = 5$ and $u_4 = 3$.

Then $u_2 = 5$ and $(3,3) \in B$ force $v_3 = 7$.

Then $v_3 = 7$ and $(3,3) \in B$ force $u_3 = 1$.

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 9$, $q_{2,2} = 8$, $q_{3,1} = 6$, $q_{3,2} = 5$, $q_{4,2} = 6$ and $q_{4,3} = 4$.

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	•	4	•	16		0
		0		0		9	
2	3	•	7		2	•	5
		0		8		0	
3	13		8		6	•	1
		6		5		0	
4	5	•	7		8		3
		0		6		4	
v_j	8		4		7		

All numbers $q_{i,j}$ are non-negative for the current feasible basic solution. This solution is therefore optimal. Indeed, arguing as before we find that the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 208 + 9\overline{x}_{1,3} + 8\overline{x}_{2,2} + 6\overline{x}_{3,1} + 5\overline{x}_{3,2} + 6\overline{x}_{4,2} + 4\overline{x}_{4,3}.$$

We conclude that X is an basic optimal solution, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$