MA3484—Methods of Mathematical Economics School of Mathematics, Trinity College Hilary Term 2017 Lecture 9 (February 3, 2017)

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# 3.10. A Method for finding Basic Optimal Solutions

We continue with the study of the optimization problem introduced in the discussion of the minimum cost method.

### Example

We seek to determine non-negative real numbers  $x_{i,j}$  for

i = 1, 2, 3, 4 and j = 1, 2, 3 that minimize  $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$ , where  $c_{i,j}$  is the coefficient in the *i*th row and *i*th row are a state of *i*.

is the coefficient in the *i*th row and *j*th column of the cost matrix C, where

$$C = \left(\begin{array}{rrrrr} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{array}\right)$$

subject to the constraints

## 3. The Transportation Problem (continued)

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad (i = 1, 2, 3, 4)$$

 $\mathsf{and}$ 

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad (j = 1, 2, 3),$$

where

$$s_1 = 13$$
,  $s_2 = 8$ ,  $s_3 = 11$ ,  $s_4 = 13$ ,  
 $d_1 = 19$ ,  $d_2 = 12$ ,  $d_3 = 14$ .

We have found an initial basic feasible solution by the Minimum Cost Method. This solution satisfies  $x_{i,j} = (X)_{i,j}$  for all *i* and *j*, where

$$X = \left(\begin{array}{rrrr} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{array}\right)$$

We next determine whether this initial basic feasible solution is an optimal solution, and, if not, how to adjust the basis to obtain a solution of lower cost.

We determine  $u_1, u_2, u_3, u_4$  and  $v_1, v_2, v_3$  such that  $c_{i,j} = v_j - u_i$  for all  $(i, j) \in B$ , where B is the initial basis.

We seek a solution with  $u_1 = 0$ . We then determine  $q_{i,j}$  so that  $c_{i,j} = v_j - u_i + q_{i,j}$  for all *i* and *j*.

We therefore complete the following tableau:-

$c_{i,j} \searrow q_{i,j}$	1		2		3		ui
1	8	٠	4	٠	16		0
		0		0		?	
2	3		7		2	٠	?
		?		?		0	
3	13	٠	8		6	٠	?
		0		?		0	
4	5	٠	7		8		?
		0		?		?	
Vj	?		?		?		

Now  $u_1 = 0$ ,  $(1, 1) \in B$  and  $(1, 2) \in B$  force  $v_1 = 8$  and  $v_2 = 4$ . After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1		2		3		u <sub>i</sub>
1	8	٠	4	٠	16		0
		0		0		?	
2	3		7		2	٠	?
		?		?		0	
3	13	٠	8		6	٠	?
		0		?		0	
4	5	٠	7		8		?
		0		?		?	
Vj	8		4		?		

Then  $v_1 = 8$ ,  $(3, 1) \in B$  and  $(4, 1) \in B$  force  $u_3 = -5$  and  $u_4 = 3$ . After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1		2		3		ui
1	8	٠	4	٠	16		0
		0		0		?	
2	3		7		2	٠	?
		?		?		0	
3	13	٠	8		6	٠	-5
3	13	• 0	8	?	6	• 0	-5
3	13 5	• 0	8 7	?	6 8	• 0	-5 3
		• 0 • 0	8	? ?		• 0 ?	

Then  $u_3 = -5$  and  $(3,3) \in B$  force  $v_3 = 1$ . After entering this value the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1		2		3		u <sub>i</sub>
1	8	٠	4	٠	16		0
		0		0		?	
2	3		7		2	٠	?
		?		?		0	
3	13	٠	8		6	٠	-5
		0		?		0	
4	5	٠	7		8		3
		0		?		?	
Vj	8		4		1		

Then  $v_3 = 1$  and  $(2, 3) \in B$  force  $u_2 = -1$ .

After entering the numbers  $u_i$  and  $v_j$ , the tableau is as follows:—

$c_{i,j}\searrow q_{i,j}$	1		2		3		ui
1	8	٠	4	٠	16		0
		0		0		?	
2	3		7		2	٠	-1
		?		?		0	
3	13	٠	8		6	٠	-5
		0		?		0	
4	5	٠	7		8		3
		0		?		?	
Vj	8		4		1		

Computing the numbers  $q_{i,j}$  such that  $c_{i,j} + u_i = v_j + q_{i,j}$ , we find that  $q_{1,3} = 15$ ,  $q_{2,1} = -6$ ,  $q_{2,2} = 2$ ,  $q_{3,2} = -1$ ,  $q_{4,2} = 6$  and  $q_{4,3} = 10$ .

The completed tableau is as follows:----

$c_{i,j} \searrow q_{i,j}$	1		2		3		ui
1	8	٠	4	•	16		0
		0		0		15	
2	3		7		2	٠	-1
		-6		2		0	
3	13	٠	8		6	٠	-5
		0		-1		0	
4	5	٠	7		8		3
		0		6		10	
Vj	8		4		1		

The initial basic feasible solution is not optimal because some of the quantities  $q_{i,j}$  are negative. To see this, suppose that the numbers  $\overline{x}_{i,j}$  for i = 1, 2, 3, 4 and j = 1, 2, 3 constitute a feasible solution to the given problem. Then  $\sum_{j=1}^{3} \overline{x}_{i,j} = s_i$  for i = 1, 2, 3 and  $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$  for j = 1, 2, 3, 4. It follows that

$$\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} \overline{x}_{i,j} = \sum_{i=1}^{4} \sum_{j=1}^{3} (v_j - u_i + q_{i,j}) \overline{x}_{i,j}$$
$$= \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4} \sum_{j=1}^{3} q_{i,j} \overline{x}_{i,j}.$$

Applying this identity to the initial basic feasible solution, we find that  $\sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i = 238$ , given that 238 is the cost of the initial basic feasible solution. Thus the cost  $\overline{C}$  of any feasible solution  $(\overline{x}_{i,j})$  satisfies

$$\overline{C} = 238 + 15\overline{x}_{1,3} - 6\overline{x}_{2,1} + 2\overline{x}_{2,2} - \overline{x}_{3,2} + 6\overline{x}_{4,2} + 10\overline{x}_{4,3}.$$

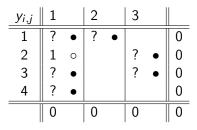
One could construct feasible solutions with  $\overline{x}_{2,1} < 0$  and  $\overline{x}_{i,j} = 0$  for  $(i,j) \notin B \cup \{(2,1)\}$ , and the cost of such feasible solutions would be lower than that of the initial basic solution. We therefore seek to bring (2,1) into the basis, removing some other element of the basis to ensure that the new basis corresponds to a feasible basic solution.

The procedure for achieving this requires us to determine a  $4 \times 3$  matrix Y satisfying the following conditions:—

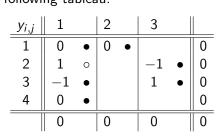
• 
$$y_{2,1} = 1;$$

- $y_{i,j} = 0$  when  $(i,j) \notin B \cup \{(2,1)\};$
- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients  $y_{i,j}$  of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—



The constraints that  $y_{2,1} = 1$ ,  $y_{i,j} = 0$  when  $(i, j) \notin B$  and the constraints requiring the rows and columns to sum to zero determine the values of  $y_{i,j}$  for all  $y_{i,j} \in B$ . These values are recorded in the following tableau:—



We now determine those values of  $\lambda$  for which  $X + \lambda Y$  is a feasible solution, where

$$X + \lambda Y = \left(egin{array}{cccc} 1 & 12 & 0 \ \lambda & 0 & 8 - \lambda \ 5 - \lambda & 0 & 6 + \lambda \ 13 & 0 & 0 \end{array}
ight).$$

In order to drive down the cost as far as possible, we should make  $\lambda$  as large as possible, subject to the requirement that all the coefficients of the above matrix should be non-negative numbers.

Accordingly we take  $\lambda = 5$ . Our new basic feasible solution X is then as follows:—

$$X = \left(\begin{array}{rrrr} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{array}\right)$$

•

We regard X as the current feasible basic solution.

The cost of the current feasible basic solution X is

$$8 \times 1 + 4 \times 12 + 3 \times 5 + 2 \times 3 + 6 \times 11$$
  
+ 5 \times 13  
= 8 + 48 + 15 + 6 + 66 + 65  
= 208.

The cost has gone down by 30, as one would expect (the reduction in the cost being  $-\lambda q_{2,1}$  where  $\lambda = 5$  and  $q_{2,1} = -6$ ).

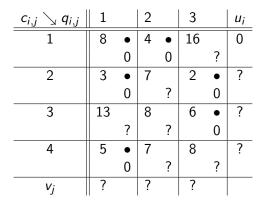
The current basic feasible solution X is associated with the basis B where

$$B = \{(1,1), (1,2), (2,1), (2,3), (3,3), (4,1)\}.$$

We now determine, for the current basis *B* values  $u_1, u_2, u_3, u_4$  and  $v_1, v_2, v_3$  such that  $c_{i,j} = v_j - u_i$  for all  $(i,j) \in B$ . the initial basis. We seek a solution with  $u_1 = 0$ . We then determine  $q_{i,j}$  so that  $c_{i,j} = v_j - u_i + q_{i,j}$  for all *i* and *j*.

### 3. The Transportation Problem (continued)

We therefore complete the following tableau:-



Now  $u_1 = 0$ ,  $(1, 1) \in B$  and  $(1, 2) \in B$  force  $v_1 = 8$  and  $v_2 = 4$ . Then  $v_1 = 8$ ,  $(2, 1) \in B$  and  $(4, 1) \in B$  force  $u_2 = 5$  and  $u_4 = 3$ . Then  $u_2 = 5$  and  $(3, 3) \in B$  force  $v_3 = 7$ . Then  $v_3 = 7$  and  $(3, 3) \in B$  force  $u_3 = 1$ .

### 3. The Transportation Problem (continued)

Computing the numbers  $q_{i,j}$  such that  $c_{i,j} + u_i = v_j + q_{i,j}$ , we find that  $q_{1,3} = 9$ ,  $q_{2,2} = 8$ ,  $q_{3,1} = 6$ ,  $q_{3,2} = 5$ ,  $q_{4,2} = 6$  and  $q_{4,3} = 4$ . The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		ui
1	8	٠	4	٠	16		0
		0		0		9	
2	3	•	7		2	٠	5
		0		8		0	
3	13		8		6	٠	1
		6		5		0	
4	5	٠	7		8		3
		0		6		4	
Vj	8		4		7		

All numbers  $q_{i,j}$  are non-negative for the current feasible basic solution. This solution is therefore optimal. Indeed, arguing as before we find that the cost  $\overline{C}$  of any feasible solution  $(\overline{x}_{i,j})$  satisfies

$$\overline{C} = 208 + 9\overline{x}_{1,3} + 8\overline{x}_{2,2} + 6\overline{x}_{3,1} + 5\overline{x}_{3,2} + 6\overline{x}_{4,2} + 4\overline{x}_{4,3}.$$

We conclude that X is an basic optimal solution, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}$$