MA3484—Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 8 (February 3, 2017)

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3.8. Formal Description of the Minimum Cost Method

We describe the *Minimum Cost Method* for finding an initial basic feasible solution to a transportation problem.

Consider a transportation problem specified by positive integers m and n and non-negative real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$, where $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, n\}$. A feasible solution consists of an array of non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in J$ with the property that $\sum_{j \in J} x_{i,j} = s_i$ for all $i \in I$ and $\sum_{i \in I} x_{i,j} = d_j$ for all $j \in J$. The objective of the problem is to find a feasible solution that minimizes cost, where the cost of a feasible solution $(x_{i,j} : i \in I \text{ and} j \in J)$ is $\sum_{i \in I} \sum_{j \in J} c_{i,j} x_{i,j}$.

In applying the Minimum Cost Method to find an initial basic solution to the Transportation we apply an algorithm that corresponds to the determination of elements $(i_1, j_1), (i_2, j_2), \ldots, (i_{m+n-1}, j_{m+n-1})$ of $I \times J$ and of subsets $I_0, I_1, \ldots, I_{m+n-1}$ of I and $J_0, J_1, \ldots, J_{m+n-1}$ of J such that $I_0 = I$, $J_0 = J$, and such that, for each integer k between 1 and $m + n - 1$, exactly one of the following two conditions is satisfied:—

- **(i)** $i_k \notin I_k$, $j_k \in J_k$, $I_{k-1} = I_k \cup \{i_k\}$ and $J_{k-1} = J_k$;
- **(ii)** $i_k \in I_k$, $j_k \notin J_k$, $I_{k-1} = I_k$ and $J_{k-1} = J_k \cup \{j_k\}$;

Indeed let $I_0 = I$, $J_0 = J$ and $B_0 = \{0\}$. The Minimum Cost Method algorithm is accomplished in $m + n - 1$ stages.

Let k be an integer satisfying $1 \leq k \leq m + n - 1$ and that subsets I_{k-1} of I, J_{k-1} of J and B_{k-1} of $I \times J$ have been determined in accordance with the rules that apply at previous stages of the Minimum Cost algorithm. Suppose also that non-negative real numbers $x_{i,j}$ have been determined for all ordered pairs (i, j) in $I \times J$ that satisfy either $i \notin I_{k-1}$ or $j \notin J_{k-1}$ so as to satisfy the following conditions:—

- $\displaystyle\sum_{j \in J \setminus J_{k-1}} x_{i,j} \leq s_i$ whenever $i \in I_{k-1}$;
- $\displaystyle\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_{k-1}$;
- $\displaystyle\sum_{i \in I \setminus I_{k-1}} x_{i,j} \leq d_j$ whenever $j \in J_{k-1}$;
- $\displaystyle\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_{k-1}$.

The Minimum Cost Method specifies that one should choose $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ so that

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1},$$

and set $B_k = B_{k-1} \cup \{(i_k, j_k)\}$. Having chosen (i_k, j_k), the non-negative real number x_{i_k,j_k} is then determined so that

$$x_{i_k,j_k} = \min \left(s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j}, \;\; d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k} \right).$$

The subsets I_k and J_k of I and J respectively are then determined, along with appropriate values of $x_{i,j}$, according to the following rules:—

(i) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} < d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $I_k = I_{k-1} \setminus \{i_k\}$ and $J_k = J_{k-1}$, and we also let $x_{i_k,j} = 0$ for all $j \in J_{k-1} \setminus \{j_k\}$;

(ii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} > d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $J_k = J_{k-1} \setminus \{j_k\}$ and $I_k = I_{k-1}$, and we also let $x_{i,j_k} = 0$ for all $i \in I_{k-1} \setminus \{i_k\}$;

(iii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} = d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we determine I_k and J_k and the corresponding values of $x_{i,j}$ either in accordance with the specification in rule (i) above or else in accordance with the specification in rule (ii) above.

These rules ensure that the real numbers $x_{i,j}$ determined at this stage are all non-negative, and that the following conditions are satisfied at the conclusion of the kth stage of the Minimum Cost Method algorithm:—

- $\displaystyle\sum_{j \in J \setminus J_k} x_{i,j} \leq s_i$ whenever $i \in I_k$;
- $\displaystyle\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_k$;
- $\displaystyle\sum_{i \in I \setminus I_k} x_{i,j} \leq d_j$ whenever $j \in J_k$;
- $\displaystyle\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_k$.

3. The Transportation Problem (continued)

At the completion of the final stage (for which $k = m + n - 1$) we have determined a subset B of $I \times J$, where $B = B_{m+n-1}$, together with non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in I$ that constitute a feasible solution to the given transportation problem.

3.9. Formal Description of the Northwest Corner Method

The *Northwest Corner Method* for finding a basic feasible solution proceeds according to the stages of the *Minimum Cost Method* above, differing only from that method in the choice of the ordered pair (i_k, j_k) at the kth stage of the method. In the Minimum Cost Method, the ordered pair (i_k, j_k) is chosen such that $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ and

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1}$$

(where the sets I_{k-1}, J_{k-1} are determined as in the specification of the Minimum Cost Method).

In applying the Northwest Corner Method, costs associated with ordered pairs (i, j) in $I \times J$ are not taken into account. Instead (i_k, j_k) is chosen so that i_k is the minimum of the integers in I_{k-1} and j_k is the minimum of the integers in J_{k-1}. Otherwise the specification of the Northwest Corner Method corresponds to that of the Minimum Cost Method, and results in a basic feasible solution of the given transportation problem.