MA3484—Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2017
Lecture 7 (February 2, 2017)

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3.6. The Minimum Cost Method for finding Basic Feasible Solutions

We discuss another method for finding an initial basic feasible solution of a transportation problem. This method is similar to the Northwest Corner Method, but takes account of the transport costs encoded in the cost matrix. The method is known as the *Minimum Cost Method*, on account of the method of selecting the cell of the tableau to be filled in at each stage in the application of the algorithm. The initial basic feasible solution obtained by this method is not necessarily optimal.

Example

Let $c_{i,j}$ be the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

and let

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We seek to find non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$ subject to the following constraints:

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad \text{for} \quad i = 1, 2, 3, 4,$$

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad \text{for} \quad j = 1, 2, 3,$$

and $x_{i,j} \geq 0$ for all i and j.

For this problem the supply vector is $(13, 8, 11, 13)$ and the demand vector is $(19, 12, 14)$. The components of both the supply vector and the demand vector add up to 45.

In order to start the process of finding an initial basic solution for this problems, we set up a tableau that records the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem, whilst leaving cells to be filled in with the values of the non-negative real numbers $x_{i,j}$ that will specify the initial basic feasible solution. The resultant tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 ?	16 ?	13
2	3 ?	7 ?	2 ?	8
3	13 ?	8 ?	6 ?	11
4	5 ?	7 ?	8 ?	13
d_j	19	12	14	45

We apply the minimum cost method to find an initial basic solution.

The cell with lowest cost is the cell $(2, 3)$. We assign to this cell the maximum value possible, which is the minimum of s_2, which is 8, and d_3, which is 14. Thus we set $x_{2,3} = 8$. This forces $x_{2,1} = 0$ and $x_{2,2} = 0$. The pair $(2, 3)$ is added to the current basis.

3. The Transportation Problem (continued)

At the completion of the first stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 ?	16 ?	13
2	3 0	7 0	2 • 8	8
3	13 ?	8 ?	6 ?	11
4	5 ?	7 ?	8 ?	13
d_j	19	12	14	45

We enter a • symbol into the tableau in the relevant cell to indicate that $(1, 2)$ will be belong to the basis constructed by this method.

3. The Transportation Problem (continued)

The next undetermined cell of lowest cost is $(1,2)$. We assign to this cell the minimum of s_1, which is 13, and $d_2 - x_{2,2}$, which is 12. Thus we set $x_{1,2} = 12$. This forces $x_{3,2} = 0$ and $x_{4,2} = 0$. The pair $(1,2)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 $\bullet$ 12	16 ?	13
2	3 0	7 0	2 $\bullet$ 8	8
3	13 ?	8 0	6 ?	11
4	5 ?	7 0	8 ?	13
d_j	19	12	14	45

The next undetermined cell of lowest cost is $(4,1)$. We assign to this cell the minimum of $s_4 - x_{4,2}$, which is 13, and $d_1 - x_{2,1}$, which is 19. Thus we set $x_{4,1} = 13$. This forces $x_{4,3} = 0$. The pair $(4,1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1		2		3		s_i
1	8		4	•	16		
		?		12		?	13
2	3		7		2	•	
		0		0		8	8
3	13		8		6		
		?		0		?	11
4	5	•	7		8		
		13		0		0	13
d_j		19		12		14	45

The next undetermined cell of lowest cost is $(3,3)$. We assign to this cell the minimum of $s_3 - x_{3,2}$, which is 11, and $d_3 - x_{2,3} - x_{4,3}$, which is 6 $(= 14 - 8)$. Thus we set $x_{3,3} = 6$. This forces $x_{1,3} = 0$. The pair $(3,3)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1		2		3		s_i
1	8		4	•	16		
		?		12		0	13
2	3		7		2	•	
		0		0		8	8
3	13		8		6	•	
		?		0		6	11
4	5	•	7		8		
		13		0		0	13
d_j		19		12		14	45

The next undetermined cell of lowest cost is $(1, 1)$. We assign to this cell the minimum of $s_1 - x_{1,2} - x_{1,3}$, which is 1, and $d_1 - x_{2,1} - x_{4,1}$, which is 6. Thus we set $x_{1,1} = 1$. The pair $(1, 1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1		2		3		s_i
1	8	•	4	•	16		
		1		12		0	13
2	3		7		2	•	
		0		0		8	8
3	13		8		6	•	
		?		0		6	11
4	5	•	7		8		
		13		0		0	13
d_j		19		12		14	45

The final undetermined cell is $(3,1)$. We assign to this cell the common value of $s_3 - x_{3,2} - x_{3,3}$ and $d_1 - x_{1,1} - x_{2,1} - x_{4,1}$, which is 5. Thus we set $x_{3,1} = 5$. The pair $(3,1)$ is added to the current basis. At the completion of this final stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1		2		3		s_i
1	8	•	4	•	16		
		1		12		0	13
2	3		7		2	•	
		0		0		8	8
3	13	•	8		6	•	
		5		0		6	11
4	5	•	7		8		
		13		0		0	13
d_j		19		12		14	45

The initial basis is thus B where

$$B = \{(1,1),\ \ (1,2),\ \ (2,3),\ \ (3,1),\ \ (3,3),\ \ (4,1)\}.$$

The basic feasible solution is represented by the 6×5 matrix X, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

The cost of this initial feasible basic solution is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 2 \times 8 + 13 \times 5 + 6 \times 6 \\ & \qquad + 5 \times 13 \\ = \ & 8 + 48 + 16 + 65 + 36 + 65 \\ = \ & 238. \end{aligned}$$

3.7. Effectiveness of the Minimum Cost Method

We now discuss the reasons why the Minimum Cost Method yields a feasible solution to a transportation problem that is a basic feasible solution.

Consider a transportation problem with m suppliers and n recipients, determined by a supply vector $\mathbf{s}$, a demand vector $\mathbf{d}$ and a cost matrix C, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

and where $\mathbf{d} \in \mathbb{R}^n$ and cost matrix C, We denote by $c_{i,j}$ the coefficient in the ith row and jth column of the matrix C.

The Minimum Cost Method determines a feasible solution to this transportation problem. A feasible solution is represented by an $m \times n$ matrix X whose coefficients $x_{i,j}$ satisfy the following conditions: $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. We must show that there exists a basis B such that the feasible solution determined by the Minimum Cost Method satisfies $x_{i,j} = 0$ when $(i, j) \notin B$.

In applying the Minimum Cost Method, we begin by locating a coefficient of the cost matrix which does not exceed the other coefficients of this matrix. Renumbering the suppliers and recipients, if necessary, we may assume, without loss of generality, that $c_{i,j} \geq c_{m,n}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The feasible solution with coefficients $x_{i,j}$ that results from application of the Minimum Cost Method then conforms to a structure specified in at least one of the two cases that are described immediately below:—

- in *Case I*, the following conditions are satisfied: $d_n \leq s_m$; $x_{m,n} = d_n$; $x_{i,n} = 0$ when $1 \leq i < n$; $\sum_{j=1}^{n-1} x_{i,j} = s_i$ for $1 \leq i < m$; $\sum_{j=1}^{n-1} x_{m,j} = s_m - d_n$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $1 \leq j < n$; and the coefficients $x_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j < n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.
- in *Case II*, the following conditions are satisfied: $s_m \leq d_n$; $x_{m,n} = s_m$; $x_{m,j} = 0$ when $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,j} = d_j$ for $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,n} = d_n - s_m$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $1 \leq i < m$; and the coefficients $x_{i,j}$ with $1 \leq i < m$ and $1 \leq j \leq n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.

The recursive nature of the Minimum Cost Method therefore enables us to prove that the Minimum Cost Method yields a basic feasible solution by induction on $m + n$, where m is the number of suppliers and n is the number of recipients. The Minimum Cost Method clearly yields a basic feasible solution in the trivial case where $m = n = 1$. We suppose therefore as our inductive hypothesis that the feasible solution determined by application of the Minimum Cost Method is a basic feasible solution in those cases where adding the number of suppliers to the number of recipients results in a number less than $m + n$.

In particular, we may assume that, in applying the Minimum Cost Method to the given problem with m suppliers and n recipients the matrices X' and X'' that result from application of the Minimum Cost Method to a smaller transportation problem as specified in the descriptions of *Case I* and *Case II* above.

Let us now restrict attention to *Case I*. In this case the reduced transportation is a transportation problem with m suppliers and $n-1$ recipients. The inductive hypothesis guarantees that the feasible solution that results from application of the Minimum Cost Method is a basic solution. Therefore there exists a basis B' for this reduced problem with $n+m-2$ elements, Moreover if $1 \leq i \leq m$, $1 \leq j \leq n-1$ and if $x_{i,j} \neq 0$ then $(i,j) \in B'$. The elements of the basis B' take the form of ordered pairs (i,j), where i is some integer between 1 and m and j is some integer between 1 and $n-1$. Let

$$B = B' \cup \{(m,n)\}.$$

We claim that B is a basis for a transportation problem with m suppliers and n recipients.

Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. We must show that there exist unique real numbers $z_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} z_{i,j} = a_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} z_{i,j} = b_j$ for $j = 1, 2, \ldots, n$, and $z_{i,j} = 0$ unless $(i,j) \in B$.

In particular these equations require that $\sum_{i=1}^{m} z_{i,n} = b_n$. But m is the only value of i for which $(i,n) \in B$. It follows that the coefficients $z_{i,j}$ of any basic solution determined by the basis B must satisfy $z_{i,n} = 0$ for $i < m$ and $z_{m,n} = b_n$.

It then follows that, in *Case I*, if the coefficients $z_{i,j}$ satisfy the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and if $z_{i,j} = 0$ unless $(i,j) \in B$, then these coefficients must satisfy the following conditions:—

(i) $z_{m,n} = b_n$;

(ii) $z_{i,n} = 0$ when $1 \leq i < m$;

(iii) $\displaystyle\sum_{j=1}^{n-1} z_{m,j} = a_m - b_n$

(iv) $\displaystyle\sum_{j=1}^{n-1} z_{i,j} = a_i$ when $1 \leq i < m$;

(v) $\displaystyle\sum_{i=1}^{m} z_{i,j} = b_j$ when $1 \leq j < n$.

(vi) if $j < n$ and $z_{i,j} \neq 0$ then $(i,j) \in B'$.

Now B' is a basis for a transportation problem with m suppliers and $n-1$ recipients. It follows that there exist unique real numbers $z_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j < n$ that satisfy conditions (iii), (iv), (v) and (vi) above.

It follows from this that if the numbers $z_{i,n}$ are determined in accordance with conditions (i) and (ii) above then the numbers $z_{i,j}$ are the unique real numbers that solve the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and that also satisfy $z_{i,j} = 0$ whenever $(i,j) \notin B$.

We conclude that, when the Minimum Cost Method proceeds so as to produce a feasible solution to a transportation problem with m suppliers and n recipients that conforms to the conditions specified in *Case I* above, then that feasible solution is a basic feasible solution with associated basis B. A similar argument applies when the feasible solution conforms to the conditions specified in *Case II* above. The feasible solution produced by the Minimum Cost Method conforms to conditions specified in one or other of these two cases. We conclude therefore that the Minimum Cost Method always determines a basic feasible solution to a transportation problem.