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Proposition 3.7

Given any feasible solution of a transportation problem, there exists a basic feasible solution with whose cost does not exceed that of the given solution.

Proof

Let m and n be positive integers, and let let the $m \times n$ matrix X represent a feasible solution of a transportation problem with supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C, where C is an $m \times n$ matrix with real coefficients. Then $s_i \geq 0$ for $i = 1, 2, \ldots, m$ and $d_j \geq 0$ for $j = 1, 2, \ldots, n$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

Also $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. The cost of the feasible solution X is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C.

If the feasible solution X is itself basic then there is nothing to prove. Suppose therefore that X is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because X is not a basic solution to the Transportation Problem, there does not exist any basis B for the transportation problem satisfying $K \subset B$. It therefore follows from Proposition 3.6 that there exists a non-zero $m \times n$ matrix Y whose coefficients $y_{i,j}$ satisfy the following conditions:—

- $\displaystyle\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\displaystyle\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ when $(i, j) \notin K$.

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} y_{i,j} \geq 0$, where the quantities $c_{i,j}$ are the coefficients of the cost matrix C, because otherwise we can replace Y with $-Y$.

Let $Z_\lambda = X - \lambda Y$ for all real numbers λ, and let $z_{i,j}(\lambda)$ denote the coefficient $(Z_\lambda)_{i,j}$ in the ith row and jth column of the matrix Z_λ. Then $z_{i,j}(\lambda) = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover

- $\sum_{j=1}^{n} z_{i,j}(\lambda) = s_i$;
- $\sum_{i=1}^{m} z_{i,j}(\lambda) = d_j$;
- $z_{i,j}(\lambda) = 0$ whenever $(i, j) \notin K$;
- $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} z_{i,j}(\lambda) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix Y is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair (i,j) belonging to the set K for which $y_{i,j} > 0$. Let

$$\lambda_0 = \mathrm{minimum}\left\{\frac{x_{i,j}}{y_{i,j}} : (i,j) \in K \text{ and } y_{i,j} > 0\right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element (i_0, j_0) of K for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus Z_{λ_0} is a feasible solution of the given transportation problem whose cost does not exceed that of the given feasible solution X. Moreover Z_{λ_0} has fewer non-zero components than the given feasible solution X.

If Z_{λ_0} is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution. $\blacksquare$

A transportation problem has only finitely many basic feasible solutions. Indeed there are only finitely many bases for the problem, and any basis is associated with at most one basic feasible solution. Therefore there exists a basic feasible solution whose cost does not exceed the cost of any other basic feasible solution. It then follows from Proposition 3.7 that the cost of this basic feasible solution cannot exceed the cost of any other feasible solution of the given transportation problem. This basic feasible solution is thus a basic optimal solution of the Transportation Problem.

The transportation problem determined by the supply vector, demand vector and cost matrix has only finitely many basic feasible solutions, because there are only finitely many bases for the problem, and each basis can determine at most one basic feasible solution. Nevertheless the number of basic feasible solutions may be quite large.

But it can be shown that a transportation problem always has a basic optimal solution. It can be found using an algorithm that implements the Simplex Method devised by George B. Dantzig in the 1940s. This algorithm involves passing from one basis to another, lowering the cost at each stage, until one eventually finds a basis that can be shown to determine a basic optimal solution of the transportation problem.

3.5. The Northwest Corner Method

Example

We discuss in detail how to find an initial basic feasible solution of a transportation problem with 4 suppliers and 5 recipients, using a method known as the *Northwest Corner Method*. This method does not make use of cost information.

The course of the calculation is determined by the supply vector **s** and the demand vector **d**, where

$$\mathbf{s} = (9, 11, 4, 5), \quad \mathbf{d} = (6, 7, 5, 3, 8).$$

We need to fill in the entries in a tableau of the form

$x_{i,j}$	1	2	3	4	5	s_i
1	·	·	·	·	·	9
2	·	·	·	·	·	11
3	·	·	·	·	·	4
4	·	·	·	·	·	5
d_j	6	7	5	3	8	29

In the tableau just presented the labels on the left hand side identify the suppliers, the labels at the top identify the recipients, the numbers on the right hand side list the number of units that the relevant supplier must provide, and the numbers at the bottom identify the number of units that the relevant recipient must obtain. Number in the bottom right hand corner gives the common value of the total supply and the total demand.

The values in the individual cells must be non-zero, the rows must sum to the value on the left, and the columns must sum to the value on the bottom.

The Northwest Corner Method is applied recursively. At each stage the undetermined cell in at the top left (the northwest corner) is given the maximum possible value allowable with the constraints. The remainder of either the first row or the first column must then be completed with zeros. This leads to a reduced tableau to be determined with either one fewer row or else one fewer column. One continues in this fashion, as exemplified in the solution of this particular problem, until the entire tableau has been completed.

The method will also determine a basis associated with the basic feasible solution determined by the Northwest Corner Method. This basis lists the cells that play the role of northwest corner at each stage of the method.

At the first stage, the northwest corner cell is associated with supplier 1 and recipient 1. This cell is assigned a value equal to the mimimum of the corresponding column and row sums. Thus, this example, the northwest corner cell, is given the value 6, which is the desired column sum. The remaining cells in that row are given the value 0.

The tableau then takes the following form:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	·	·	·	·	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The ordered pair $(1, 1)$ commences the list of elements making up the associated basis.

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—

$x_{i,j}$	2	3	4	5	s_i
1	·	·	·	·	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The required value for the first row sum of the reduced tableau has been reduced to reflect the fact that the values in the remaining undetermined cells of the first row must sum to the value 3.

The value 3 is then assigned to the northwest corner cell of the reduced tableau (as 3 is the maximum possible value for this cell subject to the constraints on row and column sums). The reduced tableau therefore takes the following form after the second stage:—

$x_{i,j}$	2	3	4	5	s_i
1	3	0	0	0	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The main tableau at the completion of the second stage then stands as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The list of ordered pairs representing the basis elements determined at the second stage then stands as follows:—

Basis: $(1,1),(2,1),\ldots.$

The reduced tableau for the third stage then stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	4	5	3	8	20

Accordingly the northwest corner of the reduced tableau should be assigned the value 4, and the remaining elements of the first column should be assigned the value 0.

The reduced tableau at the completion of the third stage stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	4	·	·	·	11
3	0	·	·	·	4
4	0	·	·	·	5
d_j	4	5	3	8	20

The main tableau and list of basis elements at the completion of the third stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	·	·	·	11
3	0	0	·	·	·	4
4	0	0	·	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),\ldots.$

The reduced tableau at the completion of the fourth stage is as follows:—

$x_{i,j}$	3	4	5	s_i
2	5	·	·	7
3	0	·	·	4
4	0	·	·	5
d_j	5	3	8	16

The main tableau and list of basis elements at the completion of the fourth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	·	·	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),(2,3),\ldots$.

At the fifth stage the sum of the undetermined cells for the 2nd supplier must sum to 2. Therefore the main tableau and list of basis elements at the completion of the fifth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1), (2,1), (2,2), (2,3), (2,4), \ldots.$

At the sixth stage the sum of the undetermined cells for the 4th recipient must sum to 1. Therefore the main tableau and list of basis elements at the completion of the sixth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	·	4
4	0	0	0	0	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),(2,3),(2,4),(3,4),\ldots.$

Two further stages suffice to complete the tableau. Moreover, at the completion of the eighth and final stage the main tableau and list of basis elements stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)$.

3. The Transportation Problem (continued)

We now check that we have indeed obtained a basis B, where

$$B = \{(1,1),(2,1),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\}.$$

If B is indeed a basis, then arbitrary values s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5 should determine corresponding values of $x_{i,j}$ for $(i,j) \in B$, as indicated in the following tableau:—

$x_{i,j}$	1	2	3	4	5	
1	$x_{1,1}$	$x_{1,2}$				s_1
2		$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		s_2
3				$x_{3,4}$	$x_{3,5}$	s_3
4					$x_{4,5}$	s_4
	d_1	d_2	d_3	d_4	d_5	

Now analysis of the Northwest Corner Method shows that, when successive elements of the set B are ordered by the stage of the method at which they are determined. Then the value of $x_{i',j'}$ for a given ordered pair $(i', j') \in B$ is determined by the values of the row sums s_i, the column sums d_j, together with the values $x_{i,j}$ for the ordered pairs (i, j) in the set B determined at earlier stages of the method.

In the specific numerical example that we have just considered, we find that the values of $x_{i,j}$ for ordered pairs (i,j) in the set B, where

$$B = \{(1,1), (2,1), (2,2), (2,3), (2,4), (3,4), (3,5), (4,5)\},$$

are determined by solving, successively, the following equations:—

$$x_{1,1} = d_1, \quad x_{1,2} = s_1 - x_{1,1}, \quad x_{2,2} = d_2 - x_{1,2},$$

$$x_{2,3} = d_3, \quad x_{2,4} = s_2 - x_{2,3} - x_{2,2}, \quad x_{3,4} = d_4 - x_{2,4},$$

$$x_{3,5} = s_3 - x_{3,4}, \quad x_{4,5} = d_5 - x_{3,5},$$

It follows that the values of $x_{i,j}$ for $(i,j) \in B$ are indeed determined by s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5.

Indeed we find that

$$
\begin{aligned}
x_{1,1} &= d_1, \\
x_{1,2} &= s_1 - d_1, \\
x_{2,2} &= d_2 - s_1 + d_1, \\
x_{2,3} &= d_3, \\
x_{2,4} &= s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{3,4} &= d_4 - s_2 + d_3 + d_2 - s_1 + d_1, \\
x_{3,5} &= s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{4,5} &= d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1.
\end{aligned}
$$

Note that, in this specific example, the values of $x_{i,j}$ for ordered pairs (i,j) in the basis B are expressed as sums of terms of the form $\pm s_i$ and $\pm d_j$. Moreover the summands s_i all have the same sign, the summands d_j all have the same sign, and the sign of the terms s_i is opposite to the sign of the terms d_j. Thus, for example

$$x_{4,5} = (d_1 + d_2 + d_3 + d_4 + d_5) - (s_1 + s_2 + s_3).$$

This pattern is in fact a manifestation of a general result applicable to all instances of the Transportation Problem.

Remark

The basic feasible solution produced by applying the Northwest Corner Method is just one amongst many basic feasible solutions. There are many others. Some of these may be obtained on applying the Northwest Corner Method after reordering the rows and columns (thus renumbering the suppliers and recipients).

It would take significant work to calculate all basic feasible solutions and then calculate the cost associated with each one.