MA3484—Methods of Mathematical Economics
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Proposition 3.5

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers, and let K be a subset of $I \times J$. Suppose that, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists an $m \times n$ matrix X with real coefficients belonging to M_K with the following properties:

(i) $\sum_{j=1}^{n} (X)_{i,j} = y_i$ *for* $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = z_j$ *for* $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ *unless* $(i, j) \in K$.

Then there exists a basis B for the transportation problem satisfying $B \subset K$.

Proof

First we define bases for the vector spaces involved in the proof. For each integer i between 1 and m, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer j between 1 and n, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

For each ordered pair (i, j) of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)} \in M_n(\mathbb{R})$ be defined such that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let M_K denote the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m.n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\},$$

let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and let $\theta_K \colon M_K \to W$ be the linear transformation defined so that $\theta_K(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$, where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$.

Then

$$X = \sum_{(i,j)\in K} (X)_{i,j} E^{(i,j)}$$

for all $X \in M_K$, and therefore

$$\theta_K(X) = \sum_{(i,j)\in K} (X)_{i,j}\theta(E^{(i,j)}) = \sum_{(i,j)\in K} (X)_{i,j}(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$$

for all $X \in M_K$. The conditions of the proposition ensure that that the ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ of basis vectors for which (i, j) belongs to K span the vector space W. It then follows from standard linear algebra that there exists a subset B of K such that those ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which (i, j) belongs to B constitute a basis for the vector space W (see Corollary 2.3).

Thus, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to W, there exist uniquely determined real numbers $x_{i,j}$ for all $(i,j) \in B$ such that

$$(\mathbf{y}, \mathbf{z}) = \sum_{(i,j)\in B} x_{i,j}(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}).$$

Let $X \in M_B$ be the $m \times n$ matrix defined such that $(X)_{i,j} = x_{i,j}$ for all $(i,j) \in B$ and $(X)_{i,j} = 0$ for all $(i,j) \in (I \times J) \setminus B$. Then X is the unique $m \times n$ matrix with the properties that $\rho(X) = \mathbf{y}$, $\sigma(X) = \mathbf{z}$ and $X_{(i,j)} = 0$ unless $(i,j) \in B$. It follows that the subset B of K is the required basis for the transportation problem. ∎

Proposition 3.6

Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be a subset of $I \times J$. Suppose that there is no basis B of the transportation problem for which $K \subset B$. Then there exists a non-zero $m \times n$ matrix Y with real coefficients which satisfies the following conditions:

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ *for* $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ *for* $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ *when* $(i, j) \notin K$.

Proof

For each integer i between 1 and m, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer j between 1 and n, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases},$$

and let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Now follows from Proposition 2.2 that if the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in K$ were linearly independent then there would exist a subset B of $I \times J$ satisfying $K \subset B$ such that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in B$ would constitute a basis of W. It would then follow that, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to W, there would exist a unique $m \times n$ matrix X with real coefficients with the properties that $\sum_{j=1}^{m} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{n} (X)_{i,j} = (\mathbf{z})_i$ for $j = 1, 2, \ldots, n$, and $(X)_{i,j} = 0$ unless $(i,j) \in B$. The subset B of $I \times J$ would thus be a basis for the transportation problem. But the subset K is not contained in any basis for the Transportation Problem. It follows that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in K$ must be linearly dependent. Therefore there exists a non-zero $m \times n$ matrix Y with real coefficients such that $(Y)_{i,j} = 0$ when $(i,j) \notin K$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}) = (\mathbf{0}, \mathbf{0}).$$

But then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \mathbf{e}^{(i)} = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \hat{\mathbf{e}}^{(j)} = \mathbf{0},$$

and therefore

$$\sum_{j=1}^{n} (Y)_{i,j} = 0 \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} (Y)_{i,j} = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

Also $(Y)_{i,j} = 0$ unless $(i, j) \in K$. The result follows. ∎

3.4. Basic Feasible Solutions of Transportation Problems

Consider the transportation problem with m suppliers and n recipients, where the ith supplier can provide at most s_i units of some given commodity, where $s_i \geq 0$, and the jth recipient requires at least d_j units of that commodity, where $d_j \geq 0$. We suppose also that total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

The cost of transporting the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

Definition

A feasible solution $(x_{i,j})$ of a transportation problem is said to be *basic* if there exists a basis B for that transportation problem such that $x_{i,j} = 0$ whenever $(i,j) \notin B$.

Example

Consider a transportation problem where $m = n = 2$, $s_1 = 8$, $s_2 = 3$, $d_1 = 2$, $d_2 = 9$, $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. A feasible solution takes the form of a 2×2 matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with non-negative components which satisfies the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}.$$

A basic feasible solution will have at least one component equal to zero. There are four matrices with at least one zero component which satisfy the required equations. They are the following:—

$$\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 & 0 \\ -6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 9 \\ 3 & 0 \end{pmatrix}.$$

The first and third of these matrices have non-negative components. These two matrices represent basic feasible solutions to the problem, and moreover they are the only basic feasible solutions.

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

The cost of the basic feasible solution $\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is

$$8c_{1,2} + 2c_{2,1} + c_{2,2} = 24 + 8 + 1 = 33.$$

The cost of the basic feasible solution $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is

$$2c_{1,1} + 6c_{1,2} + 3c_{2,2} = 4 + 18 + 3 = 25.$$

Now any 2×2 matrix $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ satisfying the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$$

for some real number λ.

But the matrix $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ has non-negative components if and only if $0 \leq \lambda \leq 2$. It follows that the set of feasible solutions of this instance of the transportation problem is

$$\left\{ \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix} : \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 2 \right\}.$$

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. Therefore, for each real number λ satisfying $0 \leq \lambda \leq 2$, the cost $f(\lambda)$ of the feasible solution $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ is given by

$$f(\lambda) = 2\lambda + 3(8-\lambda) + 4(2-\lambda) + (1+\lambda) = 33 - 4\lambda.$$

Cost is minimized when $\lambda = 2$, and thus $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is the optimal solution of this transportation problem. The cost of this optimal solution is 25.