# MA3484—Methods of Mathematical Economics
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# 3. The Transportation Problem

## 3.1. The General Transportation Problem

The Transportation Problem can be expressed in the following form. Some commodity is supplied by $m$ suppliers and is transported from those suppliers to $n$ recipients. The $i$th supplier can supply at most $s_i$ units of the commodity, and the $j$th recipient requires at least $d_j$ units of the commodity. The cost of transporting a unit of the commodity from the $i$th supplier to the $j$th recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the $i$th supplier to the $j$th recipient.

The Transportation Problem can then be presented as follows:

> *determine* $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j} x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* $i$ *and* $j$, $\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where* $s_i \geq 0$ *for all* $i$, $d_j \geq 0$ *for all* $i$, *and* $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

The quantities $(s_1, s_2, \ldots, s_m)$ representing the quantities of the transported commodity supplied by the suppliers are the components of an $m$-dimensional vector $(s_1, s_2, \ldots, s_m)$. We refer to this vector as the *supply vector* for the transportation problem.

The quantities $(d_1, d_2, \ldots, d_m)$ representing the quantities of the transported commodity demanded by the recipients are the components of an $n$-dimensional vector $(d_1, d_2, \ldots, d_n)$. We refer to this vector as the *demand vector* for the transportation problem.

The quantities $c_{i,j}$ that represent the cost of transporting the commodity from the $i$th supplier to the $j$th recipient are the components of an $m \times n$ matrix. We refer to this matrix as the *cost matrix* for the transportation problem.

## 3.2. Transportation Problems where Supply equals Demand

Consider a transportation problem with $m$ suppliers and $n$ recipients. The following proposition shows that a solution to the transportation problem can only exist if total supply of the relevant commodity exceeds total demand for that commodity.

**Proposition 3.1**

*Let $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ be non-negative real numbers. Suppose that there exist non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ that satisfy the inequalities*

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \text{and} \quad \sum_{i=1}^{m} x_{i,j} \geq d_j.$$

Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} s_i.$$

Moreover if it is the case that

$$\sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i.$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

**Proof**

The inequalities satisfied by the non-negative real numbers $x_{i,j}$ ensure that

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

Now $s_i - \sum_{j=1}^{n} x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$. It follows that if $s_i > \sum_{j=1}^{n} x_{i,j}$ for at least one value of $i$ then $\sum_{i=1}^{m} s_i > \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}$.

Similarly $\sum_{i=1}^{m} x_{i,j} - d_j \geq 0$ for $j = 1, 2, \ldots, n$. It follows that if it is the case that $\sum_{i=1}^{m} x_{i,j} > d_j$ for at least one value of $j$ then $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n,$$

as required. ∎

We analyse the Transportation Problem in the case where total supply equals total demand. The optimization problem in this case can then be stated as follows:—

*determine* $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j} x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* $i$ *and* $j$, $\sum_{j=1}^{n} x_{i,j} = s_i$ *and* $\sum_{i=1}^{m} x_{i,j} = d_j$, *where* $s_i \geq 0$ *and* $d_j \geq 0$ *for all* $i$ *and* $j$, *and* $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

**Definition**

A *feasible* solution to a transportation problem (with equality of total supply and total demand) is represented by real numbers $x_{i,j}$, where

- $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- $\sum_{j=1}^{n} x_{i,j} = s_i$ for $= 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

**Definition**

A feasible solution $(x_{i,j})$ of a transportation problem is said to be *optimal* if it minimizes cost amongst all feasible solutions of that transportation problem.

## 3.3. Bases for the Transportation Problem

**Definition**

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where $m$ and $n$ are positive integers. Then a subset $B$ of $I \times J$ is said to be a *basis* for the transportation problem with $m$ suppliers and $n$ recipients if, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists a unique $m \times n$ matrix $X$ with real coefficients satisfying the following properties:—

**(i)** $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$;

**(ii)** $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ for $j = 1, 2, \ldots, n$;

**(iii)** $(X)_{i,j} = 0$ unless $(i, j) \in B$.

**Lemma 3.2**

*Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where $m$ and $n$ are positive integers. and let*

$$B = \{(i, j) \in I \times J : i = m \textit{ or } j = n\}.$$

*Then $B$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients.*

**Proof**

The result can readily be verified when $m = 1$ or $n = 1$. We therefore restrict attention to cases where $m > 1$ and $n > 1$.

Let

$$B = \{(i, j) \in I \times J : i = m \text{ or } j = n\},$$

where $m > 1$ and $n > 1$.

Then, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ that satisfy $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exists a unique $m \times n$ matrix $X$ with real coefficients with all the following properties:

**(i)** $\sum_{j=1}^{n} (X)_{i,j} = y_i$ for $i = 1, 2, \ldots, m$;

**(ii)** $\sum_{i=1}^{m} (X)_{i,j} = z_j$ for $j = 1, 2, \ldots, n$;

**(iii)** $(X)_{i,j} = 0$ unless $(i, j) \in B$.

This matrix $X$ has coefficients as follows: $X_{i,j} = 0$ if $i < m$ and $j < n$; $X_{i,n} = y_i$ for $i < m$; $X_{m,j} = z_j$ for $j < n$; $X_{m,n} = w$, where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

This matrix $X$ is thus of the form

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & \dots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & y_{m-1} \\ z_1 & z_2 & \dots & z_{n-1} & w \end{pmatrix},$$

where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

It follows from the definition of bases for transportation problems that the subset $B$ of $I \times J$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients. This completes the proof. ∎

We now introduce some notation for use in discussion of the theory of transportation problems.

For each integer $i$ between 1 and $m$, let $\mathbf{e}^{(i)}$ denote the $m$-dimensional vector whose $i$th component is equal to 1 and whose other components are zero. For each integer $j$ between 1 and $n$, let $\hat{\mathbf{e}}^{(j)}$ denote the $n$-dimensional vector whose $j$th component is equal to 1 and whose other components are zero. Thus

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad (\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

Moreover $\mathbf{y} = \sum_{i=1}^{m} (\mathbf{y})_i \mathbf{e}^{(i)}$ for all $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} = \sum_{j=1}^{n} (\mathbf{y})_i \mathbf{e}^{(i)}$ for all $\mathbf{z} \in \mathbb{R}^n$.

Also, for each ordered pair $(i, j)$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)}$ denote the $m \times n$ matrix that has a single non-zero coefficient equal to 1 located in the $i$th row and $j$th column of the matrix. Thus

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Moreover

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}$$

for all $m \times n$ matrices $X$ with real coefficients.

We let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined such that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then $\rho(E^{(i,j)}) = \mathbf{e}^{(i)}$ for $i = 1, 2, \ldots, m$ and $\sigma(E^{(i,j)}) = \hat{\mathbf{e}}^{(j)}$ for $j = 1, 2, \ldots, n$.

A feasible solution of the transportation problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix $C$ is represented by an $m \times n$ matrix $X$ satisfying the following three conditions:—

- The coefficients of $X$ are all non-negative;
- $\rho(X) = \mathbf{s}$;
- $\sigma(X) = \mathbf{d}$.

The cost functional $f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ is defined so that

$$f(X) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} (X)_{i,j} = \mathrm{trace}(C^T X)$$

for all $X \in M_{m,n}(\mathbb{R})$, where $C$ is the cost matrix and $c_{i,j} = (C)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

A feasible solution $\hat{X}$ of the Transportation problem is optimal if and only if $f(\hat{X}) \leq f(X)$ for all feasible solutions $X$ of that problem.

## Lemma 3.3

*Let $X$ be an $m \times n$ matrix, let $\rho(X) \in \mathbb{R}^m$ and $\sigma(X) \in \mathbb{R}^n$ be defined so that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, and let*

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

*Then $(\rho(X), \sigma(X)) \in W$.*

**Proof**

Summing the components of the vectors $\rho(X)$ and $\sigma(X)$, we find that

$$\sum_{i=1}^{m} (\rho(X))_i = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} = \sum_{j=1}^{n} (\sigma(X))_j.$$

Thus $(\rho(X), \sigma(X)) \in W$, as required. ∎

Given a subset $K$ of $I \times J$, where $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, we denote by $M_K$ the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\}.$$

The definition of bases for transportation problems then ensures that a subset $B$ of $I \times J$ is a basis for a transportation problem with $m$ suppliers and $n$-recipients if and only if the linear transformation $\theta_B \colon M_B \to W$ is an isomorphism of vector spaces, where

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$.

**Proposition 3.4**

*A basis for a transportation problem with $m$ suppliers and $n$ recipients has $m+n-1$ elements.*

**Proof**

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ and, for all $(i,j) \in I \times J$, let $E^{(i,j)}$ denote the $m \times n$ matrix defined so that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let $B$ be a basis for the transportation problem with $m$ suppliers and $n$ recipients. Then the $m \times n$ matrices $E^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the vector space $M_B$ where

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in B\}.$$

It follows that the dimension of the vector space $M_B$ is equal to the number of elements in the basis $B$.

Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and let $\theta_B \colon M_B \to W$ be defined so that $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$, and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Now the definition of bases for transportation problems ensures that $\theta \colon M_B \to W$ is an isomorphism of vector spaces. Therefore $\dim M_B = \dim W$. It follows that any two bases for a transportation problem with $m$ suppliers and $n$ recipients have the same number of elements.

Lemma 3.2 showed that

$$\{(i, j) \in I \times J : i = m \text{ or } j = n\}$$

is a basis for a transportation problem with $m$ suppliers and $n$ recipients. This basis has $m + n - 1$ elements. It follows that $\dim W = m + n - 1$, and therefore every basis for a transportation problem with $m$ suppliers and $n$ recipients has $m + n - 1$ elements, as required. ∎