MA3484—Methods of Mathematical Economics
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1.2. A Transportation Problem concerning Dairy Produce

The *Transportation Problem* is a well-known problem and important example of a linear programming problem. Discussions of the general problem are to be found in textbooks in the following places:—

- Chapter 8 of *Linear Programming: 1 Introduction*, by George B. Danzig and Mukund N. Thapa (Springer, 1997);
- Section 18 of Chapter I of *Methods of Mathematical Economics* by Joel N. Franklin (SIAM 2002).

We discuss an example of the Transportation Problem of Linear Programming, as it might be applied to optimize transportation costs in the dairy industry.

A food business has milk-processing plants located in various towns in a small country. We shall refer to these plants as *dairies*. Raw milk is supplied by numerous farmers with farms located throughout that country, and is transported by milk tanker from the farms to the dairies. The problem is to determine the catchment areas of the dairies so as to minimize transport costs.

We suppose that there are m farms, labelled by integers from 1 to m that supply milk to n dairies, labelled by integers from 1 to n. Suppose that, in a given year, the ith farm has the capacity to produce and supply a s_i litres of milk for $i = 1, 2, \ldots, n$, and that the jth dairy needs to receive at least d_j litres of milk for $j = 1, 2, \ldots, n$ to satisfy the business obligations.

The quantity $\sum_{i=1}^{m} s_i$ then represents that *total supply* of milk, and the quantity $\sum_{j=1}^{n} d_j$ represents the *total demand* for milk.

We suppose that $x_{i,j}$ litres of milk are to be transported from the ith farm to the jth dairy, and that $c_{i,j}$ represents the cost per litre of transporting this milk.

Then the total cost of transporting milk from the farms to the dairies is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}.$$

The quantities $x_{i,j}$ of milk to be transported from the farms to the dairies should then be determined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as to minimize the total cost of transporting milk.

However the ith farm can supply no more than s_i litres of milk in a given year, and that jth dairy requires at least d_j litres of milk in that year. It follows that the quantities $x_{i,j}$ of milk to be transported between farms and dairy are constrained by the requirements that

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} \geq d_j \quad \text{for } j = 1, 2, \ldots, n.$$

Suppose that the requirements of supply and demand are satisfied. Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

If it is the case that $\sum\limits_{j=1}^{n} x_{i,j} < s_i$ for at least one value of i then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} < \sum_{i=1}^{m} s_i$. Similarly if it is the case that $\sum\limits_{i=1}^{m} x_{i,j} > d_j$ for at least one value of j then $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

The following report, published in 2006, describes a study of milk transport costs in the Irish dairy industry:

> Quinlan C., Enright P., Keane M., O'Connor D. 2006. *The Milk Transport Cost Implications of Alternative Dairy Factory Location.* Agribusiness Discussion Paper No. 47. Dept of Food Business and Development. University College, Cork.

The report is available at the following URL

> http://www.ucc.ie/en/media/academic/
> foodbusinessanddevelopment/paper47.pdf

The problem was investigated using commercial software that implements standard linear programming algorithms for the solution of forms of the Transportation Problem.

The description of the methodology used in the study begins as follows:

> A transportation model based on linear programming was developed and applied the Irish dairy industry to meet the study objectives. In such transportation models, transportation costs are treated as a direct linear function of the number of units shipped. The major assumptions are:
>
> 1. The items to be shipped are homogenous (i.e., they are the same regardless of their source or destination.
> 2. The shipping cost per unit is the same regardless of the number of units shipped.
> 3. There is only one route or mode of transportation being used between each source and each destination, Stevenson, (1993).

Sources and Destinations

In 2004 there were about 25,000 dairy farmers in the Irish Republic. Hence identifying the location and size of each individual dairy farm as sources for the transportation model was beyond available resources. An alternative approach based on rural districts was adopted. There are 156 rural districts in the state and data for dairy cow numbers by rural district from the most recent livestock census was available from the Central Statistics Office (CSO). These data were converted to milk equivalent terms using average milk yield estimates.

Typical seasonal milk supply patterns were also assumed. In this way an estimate of milk availability throughout the year by rural district was derived and this could then be further converted to milk tanker loads, depending on milk tanker size.

The following is quoted from the conclusions of that report:—

> A major report on the strategic development of the Irish dairy-processing sector proposed processing plant rationalization, Strategic Development Plan for the Irish Dairy Processing Sector Prospectus, (2003). It was recommended that in the long term the number of plants processing butter, milk powder, casein and whey products in Ireland should be reduced to create four major sites for these products, with a limited number of additional sites for cheese and other products. It was estimated that savings from processing plant economies of scale would amount to €20m per annum, Prospectus (2003).

However, there is an inverse relationship between milk transport costs and plant size. Thus the optimum organisation of the industry involves a balancing of decreasing average plant costs against the increasing transport costs. In this analysis, the assumed current industry structure of 23 plants was reduced in a transportation modelling exercise firstly to 12 plants, then 9 plants and finally 6 plants and the increase in total annual milk transport costs for each alternative was calculated. Both a good location and a poor location 6 plant option were considered. The estimated milk transport costs for the different alternatives were; 4.60 cent per gallon for 23 plants; 4.85 cent per gallon for 12 plants; 5.04 cent per gallon for 9 plants; 5.24 cent per gallon for 6 plants (good location) and 5.75 cent per gallon (poor location) respectively.

In aggregate terms the results showed that milk transport costs would increase by €3, €5, €7 and €13 million per annum if processing plants were reduced from 23 to 12 to 9 to 6 (good location) and 6 (poor location) respectively. As the study of processing plant rationalization did not consider cheese plant rationalization in detail, it was inferred that the estimated saving from economies of scale of €20 million per annum was associated with between 6 and 12 processing sites. Excluding the 6 plant (poor location) option, the additional milk transport cost of moving to this reduced number of sites was estimated to be of the order of *5* million per annum. This represents about 25 per cent of the estimated benefits from economies of scale arising from processing plant rationalization.

The transportation model also facilitated a comparison of current milk catchment areas of processing plants with optimal catchment areas, assuming no change in number of processing plants. It was estimated that if dairies were to collect milk on an optimal basis, there would be an 11% reduction from the current (2005) milk transport costs.

In the "benchmark" model 23 plants were required to stay open at peak to accommodate milk supply and it was initially assumed that all 23 remained open throughout the year with the same catchment areas. However, due to seasonality in milk supply, it is not essential that all 23 plants remain open outside the peak.

Two options were analysed. The first involved allowing the model to determine the least cost transport pattern outside the peak i.e. a relaxation of the constraint of fixed catchment areas throughout the year, with all plants available for milk intake. Further modest reductions in milk transport costs were realisable in this case. The second option involved keeping only the bigger plants open outside the peak period. A modest increase in milk transport costs was estimated for this option due to tankers having to travel longer distances outside the peak period.

The analysis of milk transport costs in the Irish Dairy Industry is a significant topic in the Ph.D. thesis of the first author of the 2006 report from which the preceding quotation was taken:

> Quinlan, Carrie, Brigid, 2013. *Optimisation of the food dairy coop supply chain.* PhD Thesis, University College Cork.

which is available at the following URL:

> http://cora.ucc.ie/bitstream/handle/10468/1197/QuinlanCB_PhD2013.pdf

The Transportation Problem, with equality of total supply and total demand, can be expressed generally in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most to s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem can then be presented as follows:

determine $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ *for all* i *and* j,

$\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where*

$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.