MA3484 Methods of Mathematical Economics
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Proposition

Proposition FK-F05 *Let n be a positive integer, let I be a non-empty finite set, let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$, and, for each $i \in I$, let $\eta_i\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$. Suppose that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ with the property that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$. Then there exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$.*

Proof

We may suppose that $I = \{1, 2, \ldots, m\}$ for some positive integer m. For each $i \in I$ there exist real numbers $A_{i,1}, A_{i,2}, \ldots, A_{i,n}$ such that

$$\eta_i(v_1, v_2, \ldots, v_n) = \sum_{j=1}^{n} A_{i,j} v_j$$

for $i = 1, 2, \ldots, m$ and for all real numbers $v_1, v_2, \ldots, v_n$. Let A be the $m \times n$ matrix whose coefficient in the ith row and jth column is the real number $A_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then an n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ satisfies $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$ if and only if $A\mathbf{v} \geq \mathbf{0}$.

Farkas' Lemma (continued)

There exists an n-dimensional vector $\mathbf{c} \in \mathbb{R}^n$ such that $\varphi(\mathbf{v}) = \mathbf{c}^T\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{c}^T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $A\mathbf{v} \geq 0$. It then follows from Corollary FK-F04 that there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y} \geq \mathbf{0}$. Let $g_i = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$. Then $g_i \geq 0$ for $i = 1, 2, \ldots, m$ and $\sum_{i \in I} g_i \eta_i = \varphi$, as required. ∎

Remark

The result of Proposition FK-F05 can also be viewed as a consequence of Proposition FK-F01 applied to the convex cone in the dual space $\mathbb{R}^{n*}$ of the real vector space $\mathbb{R}^n$ generated by the linear functionals η_i for $i \in I$. Indeed let C be the subset of $\mathbb{R}^{n*}$ defined such that

$$C = \left\{ \sum_{i \in I} g_i \eta_i : g_i \geq 0 \text{ for all } i \in I \right\}.$$

It follows from Proposition FK-C02 that C is a closed convex cone in the dual space $\mathbb{R}^{n*}$ of $\mathbb{R}^n$. If the linear functional φ did not belong to this cone then it would follow from Proposition FK-F01 that there would exist a linear functional $V\colon \mathbb{R}^{n*} \to \mathbb{R}$ with the property that $V(\eta_i) \geq 0$ for all $i \in I$ and $V(\varphi) < 0$.

But given any linear functional on the dual space of a given finite-dimensional vector space, there exists some vector belonging to the given vector space such that the linear functional on the dual space evaluates elements of the dual space at that vector (see Corollary LA-16 in the discussion of linear algebra). It follows that there would exist $\mathbf{v} \in \mathbb{R}^n$ such that $V(\psi) = \psi(\mathbf{v})$ for all $\psi \in \mathbb{R}^{n*}$. But then $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$ and $\varphi(\mathbf{v}) < 0$. This contradicts the requirement that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$. To avoid this contradiction it must be the case that $\varphi \in C$, and therefore there must exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$.

Corollary

Corollary FK-F06 *Let n be a positive integer, let I be a non-empty finite set, let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$, and, for each $i \in I$, let $\eta_i\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$. Suppose that there exists a subset I_0 of I such that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ with the property that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I_0$. Then there exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$ and $g_i = 0$ when $i \notin I_0$.*

Proof

It follows directly from Proposition FK-F05 that there exist non-negative real numbers g_i for all $i \in I_0$ such that $\varphi = \sum_{i \in I_0} g_i \eta_i$.

Let $g_i = 0$ for all $i \in I \setminus I_0$. Then $\varphi = \sum_{i \in I_0} g_i \eta_i$, as required. ∎

Definition

A subset X is said to be a *convex polytope* if there exist linear functionals $\eta_1, \eta_2, \ldots, \eta_m$ on $\mathbb{R}^n$ and real numbers $s_1, s_2, \ldots, s_m$ such that

$$X = \{\mathbf{x} \in \mathbb{R}^n : \eta_i(\mathbf{x}) \geq s_i \text{ for } i = 1, 2, \ldots, m\}.$$

Optimizing Linear Functions on Convex Polytope (continued)

Let $(\eta_i : i \in I)$ be a finite collection of linear functionals on $\mathbb{R}^n$ indexed by a finite set I, let s_i be a real number for all $i \in I$, and let

$$X = \bigcap_{i \in I} \{\mathbf{x} \in \mathbb{R} : \eta_i(\mathbf{x}) \geq s_i\}.$$

Then X is a convex polytope in $\mathbb{R}^n$. A point $\mathbf{x}$ of $\mathbb{R}^n$ belongs to the convex polytope X if and only if $\eta_i(\mathbf{x}) \geq s_i$ for all $i \in I$.

Proposition

Proposition FK-07 *Let n be a positive integer, let I be a non-empty finite set, and, for each $i \in I$, let $\eta_i \colon \mathbb{R}^n \to \mathbb{R}$ be non-zero linear functional and let s_i be a real number. Let X be the convex polytope defined such that*

$$X = \bigcap_{i \in I} \{\mathbf{x} \in \mathbb{R} : \eta_i(\mathbf{x}) \geq s_i\}.$$

(Thus a point $\mathbf{x}$ of $\mathbb{R}^n$ belongs to the convex polytope X if and only if $\eta_i(\mathbf{x}) \geq s_i$ for all $i \in I$.) Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear functional on $\mathbb{R}^n$, and let $\mathbf{x}^* \in X$. Then $\varphi(\mathbf{x}^*) \leq \varphi(\mathbf{x})$ for all $\mathbf{x} \in X$ if and only if there exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$ and $g_i = 0$ whenever $\eta_i(\mathbf{x}^*) > s_i$.

Proof

Let $K = \{i \in I : \eta_i(\mathbf{x}^*) > s_i\}$. Suppose that there do not exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$ and $g_i = 0$ when $i \in K$. Corollary FK-F06 then ensures that there must exist some $\mathbf{v} \in V$ such that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I \setminus K$ and $\varphi(\mathbf{v}) < 0$. Then

$$\eta_i(\mathbf{x}^* + \lambda\mathbf{v}) = \eta_i(\mathbf{x}^*) + \lambda\eta_i(\mathbf{v}) \geq s_i$$

for all $i \in I \setminus K$ and for all $\lambda \geq 0$. If $i \in K$ then $\eta_i(\mathbf{x}^*) > s_i$. The set K is finite. It follows that there must exist some real number λ_0 satisfying $\lambda_0 > 0$ such that $\eta_i(\mathbf{x}^* + \lambda\mathbf{v}) \geq s_i$ for all $i \in K$ and for all real numbers λ satisfying $0 \leq \lambda < \lambda_0$.

Farkas' Lemma (continued)

Combining the results in the cases when $i \in I \setminus K$ and when $i \in K$, we find that $\eta_i(\mathbf{x}^* + \lambda\mathbf{v}) \geq s_i$ for all $i \in I$ and $\lambda \in [0, \eta_0]$, and therefore $\mathbf{x}^* + \lambda\mathbf{v} \in X$ for all real numbers λ satisfying $0 \leq \lambda \leq \lambda_0$. But

$$\varphi(\mathbf{x}^* + \lambda\mathbf{v}) = \varphi(\mathbf{x}^*) + \lambda\varphi(\mathbf{v}) < \varphi(\mathbf{x}^*)$$

whenever $\lambda > 0$. It follows that the linear functional φ cannot attain a minimum value in X at any point $\mathbf{x}^*$ for which either $K = I$ or for which K is a proper subset of I but there exist non-negative real numbers g_i for all $i \in I \setminus K$ such that $\varphi = \sum_{i \in I \setminus K} g_i \eta_i$. The result follows. ∎

Strong Duality

Example

Consider again the following linear programming problem in general primal form:—

find values of x_1, x_2, x_3 and x_4 so as to minimize the objective function

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to the following constraints:—

- $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$;
- $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 = b_2$;
- $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \geq b_3$;
- $x_1 \geq 0$ *and* $x_3 \geq 0$.

Farkas' Lemma (continued)

Now the constraint

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$$

can be expressed as a pair of inequality constraints as follows:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 &\geq b_1 \\ -a_{1,1}x_1 - a_{1,2}x_2 - a_{1,3}x_3 - a_{1,4}x_4 &\geq -b_1. \end{aligned}$$

Similarly the equality constraint involving b_2 can be expressed as a pair or inequality constraints.

Farkas' Lemma (continued)

Therefore the problem can be reformulated as follows:—

find values of x_1, x_2, x_3 and x_4 so as to minimize the objective function

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to the following constraints:—

- $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 \geq b_1$;
- $-a_{1,1}x_1 - a_{1,2}x_2 - a_{1,3}x_3 - a_{1,4}x_4 \geq -b_1$;
- $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 \geq b_2$;
- $-a_{2,1}x_1 - a_{2,2}x_2 - a_{2,3}x_3 - a_{2,4}x_4 \geq -b_2$;
- $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \geq b_3$;
- $x_1 \geq 0$;
- $x_3 \geq 0$.

Farkas' Lemma (continued)

Let

$$\varphi(x_1, x_2, x_3, x_4) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4,$$

and let

$$\begin{aligned}
\eta_1^+(x_1, x_2, x_3, x_4) &= a_{1,1} x_1 + a_{1,2} x_2 + a_{1,3} x_3 + a_{1,4} x_4, \\
\eta_1^-(x_1, x_2, x_3, x_4) &= -\eta_1(x_1, x_2, x_3, x_4), \\
\eta_2^+(x_1, x_2, x_3, x_4) &= a_{2,1} x_1 + a_{2,2} x_2 + a_{2,3} x_3 + a_{2,4} x_4, \\
\eta_2^-(x_1, x_2, x_3, x_4) &= -\eta_3(x_1, x_2, x_3, x_4), \\
\eta_3(x_1, x_2, x_3, x_4) &= a_{3,1} x_1 + a_{3,2} x_2 + a_{3,3} x_3 + a_{3,4} x_4, \\
\zeta_1(x_1, x_2, x_3, x_4) &= x_1, \\
\zeta_3(x_1, x_2, x_3, x_4) &= x_3,
\end{aligned}$$

Farkas' Lemma (continued)

Then (x_1, x_2, x_3, x_4) is a feasible solution to the primal problem if and only if this element of $\mathbb{R}^4$ belongs to the convex polytope X, where X is the subset of $\mathbb{R}^4$ consisting of all points $\mathbf{x}$ of $\mathbb{R}^4$ that satisfy the following constraints:—

- $\eta_1^+(\mathbf{x}) \geq b_1$;
- $\eta_1^-(\mathbf{x}) \geq -b_1$;
- $\eta_2^+(\mathbf{x}) \geq b_2$;
- $\eta_2^-(\mathbf{x}) \geq -b_2$;
- $\eta_3(\mathbf{x}) \geq b_3$;
- $\zeta_1(\mathbf{x}) \geq 0$;
- $\zeta_3(\mathbf{x}) \geq 0$.

An inequality constraint is said to be *binding* for a particular feasible solution **x** if equality holds in that constraint at the feasible solution. Thus the constraints on the values of η_1^+, η_1^-, η_2^+ and η_2^- are always binding at points of the convex polytope X, but the constraints determined by η_3, ζ_1 and ζ_3 need not be binding.

Suppose that the linear functional φ attains its minimum value at a point $\mathbf{x}^*$ of X, where $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$. It then follows from Proposition FK-07 that there exist non-negative real numbers p_1^+, p_1^-, p_2^+, p_2^-, p_3, q_1 and q_3 such that

$$p_1^+\eta_1^+ + p_1^-\eta_1^- + p_2^+\eta_2^+ + p_2^-\eta_2^- + p_3\eta_3 + q_1\zeta_1 + q_3\zeta_3 = \varphi.$$

Moreover $p_3 = 0$ if $\eta_3(\mathbf{x}^*) > b_3$, $q_1 = 0$ if $\zeta_1(\mathbf{x}^*) > 0$, and $q_3 = 0$ if $\zeta_3(\mathbf{x}^*) > 0$.

Farkas' Lemma (continued)

Now $\eta_1^- = -\eta_1^+$ and $\eta_2^- = -\eta_2^+$. It follows that

$$p_1\eta_1^+ + p_2\eta_2^+ + p_3\eta_3 + q_1\zeta_1 + q_3\zeta_3 = \varphi,$$

where $p_1 = p_1^+ - p_1^-$ and $p_2 = p_2^+ - p_2^-$. Moreover $p_3 = 0$ if $\sum_{i=1}^{4} a_{3,j}x_j^* > b_3$, $q_1 = 0$ if $x_1^* > 0$, and $q_3 = 0$ if $x_3^* > 0$.

It follows that

$$\begin{aligned} p_1 a_{1,1} + p_2 a_{2,1} + p_3 a_{3,1} &\leq c_1, \\ p_1 a_{1,2} + p_2 a_{2,2} + p_3 a_{3,2} &= c_2, \\ p_1 a_{1,3} + p_2 a_{2,3} + p_3 a_{3,3} &\leq c_3, \\ p_1 a_{1,4} + p_2 a_{2,4} + p_3 a_{3,4} &= c_4, \\ p_3 &\geq 0. \end{aligned}$$

Moreover $p_3 = 0$ if $\sum_{i=1}^{4} a_{3,j} x_j^* > b_3$, $\sum_{i=1}^{3} p_i a_{i,1} = c_1$ if $x_1^* > 0$, and $\sum_{i=1}^{3} p_i a_{i,3} = c_3$ if $x_3^* > 0$. It follows that (p_1, p_2, p_3) is a feasible solution of the dual problem to the feasible primal problem.

Farkas' Lemma (continued)

Moreover the complementary slackness conditions determined by the primal problem are satisfied. It therefore follows from the Weak Duality Theorem (Theorem DT-05) that (p_1, p_2, p_3) is an optimal solution to the dual problem.