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A Separating Hyperplane Theorem

Definition

A subset K of $\mathbb{R}^m$ is said to be *convex* if $(1-\mu)\mathbf{x} + \mu\mathbf{x}' \in K$ for all elements $\mathbf{x}$ and $\mathbf{x}'$ of K and for all real numbers μ satisfying $0 \leq \mu \leq 1$.

It follows from the above definition that a subset K of $\mathbb{R}^{\gtrdot}$ is a convex subset of $\mathbb{R}^m$ if and only if, given any two points of K, the line segment joining those two points is wholly contained in K.

Theorem

Theorem FK-CS-03 *Let m be a positive integer, let K be a closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$, where $\mathbf{b} \notin K$. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in K$ and $\varphi(\mathbf{b}) < c$.*

Proof

It follows from Lemma FK-CS-02 that there exists a point $\mathbf{g}$ of K such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in K$. Let $\mathbf{x} \in K$. Then $(1 - \lambda)\mathbf{g} + \lambda\mathbf{x} \in K$ for all real numbers λ satisfying $0 \leq \lambda \leq 1$, because the set K is convex, and therefore

$$|(1 - \lambda)\mathbf{g} + \lambda\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all real numbers λ satisfying $0 \leq \lambda \leq 1$. Now

$$(1 - \lambda)\mathbf{g} + \lambda\mathbf{x} - \mathbf{b} = \mathbf{g} - \mathbf{b} + \lambda(\mathbf{x} - \mathbf{g}).$$

Separating Hyperplane Theorem (continued)

It follows by a straightforward calculation from the definition of the Euclidean norm that

$$\begin{aligned} |\mathbf{g} - \mathbf{b}|^2 &\leq |(1-\lambda)\mathbf{g} + \lambda\mathbf{x} - \mathbf{b}|^2 \\ &= |\mathbf{g} - \mathbf{b}|^2 + 2\lambda(\mathbf{g} - \mathbf{b})^T(\mathbf{x} - \mathbf{g}) \\ &\qquad + \lambda^2|\mathbf{x} - \mathbf{g}|^2 \end{aligned}$$

for all real numbers λ satisfying $0 \leq \lambda \leq 1$. In particular, this inequality holds for all sufficiently small positive values of λ, and therefore

$$(\mathbf{g} - \mathbf{b})^T(\mathbf{x} - \mathbf{g}) \geq 0$$

for all $\mathbf{x} \in K$.

Separating Hyperplane Theorem (continued)

Let

$$\varphi(\mathbf{x}) = (\mathbf{g} - \mathbf{b})^T \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^m$. Then $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is a linear functional on $\mathbb{R}^m$, and $\varphi(\mathbf{x}) \geq \varphi(\mathbf{g})$ for all $\mathbf{x} \in K$. Moreover

$$\varphi(\mathbf{g}) - \varphi(\mathbf{b}) = |\mathbf{g} - \mathbf{b}|^2 > 0,$$

and therefore $\varphi(\mathbf{g}) > \varphi(\mathbf{b})$. It follows that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in K$, where $c = \frac{1}{2}\varphi(\mathbf{b}) + \frac{1}{2}\varphi(\mathbf{g})$, and that $\varphi(\mathbf{b}) < c$. The result follows. ∎

Convex Cones

Definition

Let m be a positive integer. A subset C of $\mathbb{R}^m$ is said to be a *convex cone* in $\mathbb{R}^m$ if $\lambda\mathbf{v} + \mu\mathbf{w} \in C$ for all $\mathbf{v}, \mathbf{w} \in C$ and for all real numbers λ and μ satisfying $\lambda \geq 0$ and $\mu \geq 0$.

Lemma

Lemma FK-CC01 *Let m be a positive integer. Then every convex cone in $\mathbb{R}^m$ is a convex subset of $\mathbb{R}^m$.*

Proof

Let C be a convex cone in $\mathbb{R}^m$ and let $\mathbf{v}, \mathbf{w} \in C$. Then $\lambda\mathbf{v} + \mu\mathbf{w} \in C$ for all non-negative real numbers λ and μ. In particular $(1 - \lambda)\mathbf{w} + \lambda\mathbf{v} \in C$. whenever $0 \leq \lambda \leq 1$, and thus the convex cone C is a convex set in $\mathbb{R}^m$, as required. ∎

Lemma

Lemma FK-CC02 *Let S be a subset of $\mathbb{R}^m$, and let C be the set of all elements of $\mathbb{R}^m$ that can be expressed as a linear combination of the form*

$$s_1\mathbf{a}^{(1)} + s_2\mathbf{a}^{(2)} + \cdots + s_n\mathbf{a}^{(n)},$$

where $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ are vectors belonging to S and $s_1, s_2, \ldots, s_n$ are non-negative real numbers. Then C is a convex cone in $\mathbb{R}^m$.

Proof

Let $\mathbf{v}$ and $\mathbf{w}$ be elements of C. Then there exist finite subsets S_1 and S_2 of S such that $\mathbf{v}$ can be expressed as a linear combination of the elements of S_1 with non-negative coefficients and $\mathbf{w}$ can be expressed as a linear combination of the elements of S_2 with non-negative coefficients. Let

$$S_1 \cup S_2 = \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\}.$$

Then there exist non-negative real numbers $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$ such that

$$\mathbf{v} = \sum_{j=1}^{n} s_j \mathbf{a}^{(j)} \quad \text{and} \quad \mathbf{w} = \sum_{j=1}^{n} t_j \mathbf{a}^{(j)}.$$

Let λ and μ be non-negative real numbers. Then

$$\lambda \mathbf{v} + \mu \mathbf{w} = \sum_{j=1}^{n} (\lambda s_j + \mu t_j) \mathbf{a}^{(j)},$$

and $\lambda s_j + \mu t_j \geq 0$ for $j = 1, 2, \ldots, n$. It follows that $\lambda \mathbf{v} + \mu \mathbf{w} \in S$, as required. ■

Proposition

Proposition FK-C02 *Let m be a positive integer, let $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)} \in \mathbb{R}^m$, and let C be the subset of $\mathbb{R}^m$ defined such that*

$$C = \left\{ \sum_{j=1}^{n} t_j \mathbf{a}^{(j)} : t_j \geq 0 \text{ for } j = 1, 2, \ldots, n \right\}.$$

Then C is a closed convex cone in $\mathbb{R}^m$.

Proof

It follows from Lemma FK-CC02 that C is a convex cone in $\mathbb{R}^m$. We must prove that this convex cone is a closed set.

The vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ span a vector subspace V of $\mathbb{R}^m$ that is isomorphic as a real vector space to $\mathbb{R}^k$ for some integer k satisfying $0 \leq k \leq m$. This vector subspace V of $\mathbb{R}^m$ is a closed subset of $\mathbb{R}^m$, and therefore any subset of V that is closed in V will also be closed in $\mathbb{R}^m$. Replacing $\mathbb{R}^m$ by $\mathbb{R}^k$, if necessary, we may assume, without loss of generality that the vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ span the vector space $\mathbb{R}^m$. Thus if A is the $m \times n$ matrix defined such that $(A)_{i,j} = (\mathbf{a}^{(j)})_i$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ then the matrix A is of rank m.

Let $\mathcal{B}$ be the collection consisting of all subsets B of $\{1, 2, \ldots, n\}$ for which the members of the set $\{\mathbf{a}^{(j)} : j \in B\}$ constitute a basis of the real vector space $\mathbb{R}^m$ and, for each $B \in \mathcal{B}$, let

$$C_B = \left\{ \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)} : s_i \geq 0 \text{ for } i = 1, 2, \ldots, m \right\},$$

where $j_1, j_2, \ldots, j_m$ are distinct and are the elements of the set B. It follows from Lemma FK-T01 that the set C_B is closed in $\mathbb{R}^m$ for all $B \in \mathbb{B}$.

Let $\mathbf{b} \in C$. The definition of C then ensures that there exists some $\mathbf{x} \in \mathbb{R}^n$ that satisfies $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Thus the problem of determining $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has a feasible solution. It follows from Theorem SM-02 that there exists a basic feasible solution to this problem, and thus there exist distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n and non-negative real numbers $s_1, s_2, \ldots, s_m$ such that $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ are linearly independent and

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

Therefore $\mathbf{b} \in C_B$ where

$$B = \{j_1, j_2, \ldots, j_m\}.$$

We have thus shown that, given any element $\mathbf{b}$ of C, there exists a subset B of $\{1, 2, \ldots, n\}$ belonging to $\mathcal{B}$ for which $\mathbf{b} \in C_B$. It follows from this that the subset C of $\mathbb{R}^m$ is the union of the closed sets C_B taken over all elements B of the finite set $\mathcal{B}$. Thus C is a finite union of closed subsets of $\mathbb{R}^m$, and is thus itself a closed subset of $\mathbb{R}^m$, as required. $\blacksquare$

Proposition

Proposition FK-F01 *Let C be a closed convex cone in $\mathbb{R}^m$ and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Suppose that $\mathbf{b} \notin C$. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in C$ and $\varphi(\mathbf{b}) < 0$.*

Proof

Suppose that $\mathbf{b} \notin C$. The cone C is a closed convex set. It follows from Theorem FK-CS-03 that there exists a linear functional $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{v}) > c$ for all $\mathbf{v} \in C$ and $\varphi(\mathbf{b}) < c$.

Now $\mathbf{0} \in C$, and $\varphi(\mathbf{0}) = 0$. It follows that $c < 0$, and therefore $\varphi(\mathbf{b}) \leq c < 0$.

Farkas' Lemma (continued)

Let $\mathbf{v} \in C$. Then $\lambda\mathbf{v} \in C$ for all real numbers λ satisfying $\lambda > 0$. It follows that $\lambda\varphi(\mathbf{v}) = \varphi(\lambda\mathbf{v}) > c$ and thus $\varphi(\mathbf{v}) > \dfrac{c}{\lambda}$ for all real numbers λ satisfying $\lambda > 0$, and therefore

$$\varphi(\mathbf{v}) \geq \lim_{\lambda \to +\infty} \frac{c}{\lambda} = 0.$$

We conclude that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in C$.

Thus $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in C$ and $\varphi(\mathbf{b}) < 0$, as required. ∎

Lemma

Lemma FK-F02 (Farkas' Lemma) *Let A be a $m \times n$ matrix with real coefficients, and let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional real vector. Then exactly one of the following two statements is true:—*

- **(i)** *there exists $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$;*
- **(ii)** *there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq \mathbf{0}$ and $\mathbf{y}^T \mathbf{b} < 0$.*

Proof

Let $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ be the vectors in $\mathbb{R}^m$ determined by the columns of the matrix A, so that $(\mathbf{a}^{(j)})_i = (A)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let

$$C = \left\{ \sum_{j=1}^{n} x_j \mathbf{a}^{(j)} : x_j \geq 0 \text{ for } j = 1, 2, \ldots, n \right\}.$$

It follows from Proposition FK-C02 that C is a closed convex cone in $\mathbb{R}^m$. Moreover

$$C = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} \geq \mathbf{0}\}.$$

Thus $\mathbf{b} \in C$ if and only if there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{b} = A\mathbf{x}$ and $\mathbf{x} \geq \mathbf{0}$. Therefore statement (i) in the statement of Farkas' Lemma is true if and only if $\mathbf{b} \in C$.

If $\mathbf{b} \notin C$ then it follows from Proposition FK-F01 that there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in C$ and $\varphi(\mathbf{b}) < 0$. Then there exists $\mathbf{y} \in \mathbb{R}^m$ with the property that $\varphi(\mathbf{v}) = \mathbf{y}^T\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^m$. Now $A\mathbf{x} \in C$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$. It follows that $\mathbf{y}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$. In particular $(\mathbf{y}^T A)_i = \mathbf{y}^T A\mathbf{e}^{(i)} \geq 0$ for $i = 1, 2, \ldots, m$, where $\mathbf{e}^{(i)}$ is the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero. Thus if $\mathbf{b} \notin C$ then there exists $\mathbf{y} \in \mathbb{R}^m$ for which $\mathbf{y}^T A \geq \mathbf{0}$ and $\mathbf{y}^T\mathbf{b} < 0$.

Conversely suppose that there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq 0$ and $\mathbf{y}^T \mathbf{b} < 0$. Then $\mathbf{y}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$, and therefore $\mathbf{y}^T \mathbf{v} \geq 0$ for all $\mathbf{v} \in C$. But $\mathbf{y}^T \mathbf{b} < 0$. It follows that $\mathbf{b} \notin C$. Thus statement (ii) in the statement of Farkas's Lemma is true if and only if $\mathbf{b} \notin C$. The result follows. ∎

Corollary

Corollary FK-F03 *Let A be a $m \times n$ matrix with real coefficients, and let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional real vector. Then exactly one of the following two statements is true:—*

- **(i)** *there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y} \geq \mathbf{0}$;*
- **(ii)** *there exists $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} \geq \mathbf{0}$ and $\mathbf{c}^T \mathbf{v} < 0$.*

Proof

It follows on applying Farkas's Lemma to the transpose of the matrix A that exactly one of the following statements is true:—

(i) there exists $\mathbf{y} \in \mathbb{R}^m$ such that $A^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$;

(ii) there exists $\mathbf{v} \in \mathbb{R}^m$ such that $\mathbf{v}^T A^T \geq \mathbf{0}$ and $\mathbf{v}^T\mathbf{c} < 0$.

But $\mathbf{v}^T\mathbf{c} = \mathbf{c}^T\mathbf{v}$. Also $A^T\mathbf{y} = \mathbf{c}$ if and only if $\mathbf{y}^T A = \mathbf{c}^T$, and $\mathbf{v}^T A^T \geq \mathbf{0}$ if and only if $A\mathbf{v} \geq \mathbf{0}$. The result follows. ∎

Corollary

Corollary FK-F04 *Let A be a $m \times n$ matrix with real coefficients, and let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional real vector. Suppose that $\mathbf{c}^T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $A\mathbf{v} \geq 0$. Then there exists some there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y} \geq \mathbf{0}$.*

Proof

Statement (ii) in the statement of is false, by assumption, and therefore statement (i) in the statement of that corollary must be true. The result follows. ∎

Proposition

Proposition FK-F05 *Let n be a positive integer, let I be a non-empty finite set, let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$, and, for each $i \in I$, let $\eta_i\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional on $\mathbb{R}^n$. Suppose that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ with the property that $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$. Then there exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$.*

Proof

We may suppose that $I = \{1, 2, \ldots, m\}$ for some positive integer m. For each $i \in I$ there exist real numbers $A_{i,1}, A_{i,2}, \ldots, A_{i,n}$ such that

$$\eta_i(v_1, v_2, \ldots, v_n) = \sum_{j=1}^{n} A_{i,j} v_j$$

for $i = 1, 2, \ldots, m$ and for all real numbers $v_1, v_2, \ldots, v_n$. Let A be the $m \times n$ matrix whose coefficient in the ith row and jth column is the real number $A_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then an n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ satisfies $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$ if and only if $A\mathbf{v} \geq \mathbf{0}$.

There exists an n-dimensional vector $\mathbf{c} \in \mathbb{R}^n$ such that $\varphi(\mathbf{v}) = \mathbf{c}^T\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{c}^T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $A\mathbf{v} \geq 0$. It then follows from Corollary FK-F04 that there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y} \geq \mathbf{0}$. Let $g_i = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$. Then $g_i \geq 0$ for $i = 1, 2, \ldots, m$ and $\sum_{i \in I} g_i \eta_i = \varphi$, as required. ∎

Remark

The result of Proposition FK-F05 can also be viewed as a consequence of Proposition FK-F01 applied to the convex cone in the dual space $\mathbb{R}^{n*}$ of the real vector space $\mathbb{R}^n$ generated by the linear functionals η_i for $i \in I$. Indeed let C be the subset of $\mathbb{R}^{n*}$ defined such that

$$C = \left\{ \sum_{i \in I} g_i \eta_i : g_i \geq 0 \text{ for all } i \in I \right\}.$$

It follows from Proposition FK-C02 that C is a closed convex cone in the dual space $\mathbb{R}^{n*}$ of $\mathbb{R}^n$. If the linear functional φ did not belong to this cone then it would follow from Proposition FK-F01 that there would exist a linear functional $V : \mathbb{R}^{n*} \to \mathbb{R}$ with the property that $V(\eta_i) \geq 0$ for all $i \in I$ and $V(\varphi) < 0$.

But given any linear functional on the dual space of a given finite-dimensional vector space, there exists some vector belonging to the given vector space such that the linear functional on the dual space evaluates elements of the dual space at that vector (see Corollary LA-16 in the discussion of linear algebra). It follows that there would exist $\mathbf{v} \in \mathbb{R}^n$ such that $V(\psi) = \psi(\mathbf{v})$ for all $\psi \in \mathbb{R}^{n*}$. But then $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$ and $\varphi(\mathbf{v}) < 0$. This contradicts the requirement that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $\eta_i(\mathbf{v}) \geq 0$ for all $i \in I$. To avoid this contradiction it must be the case that $\varphi \in C$, and therefore there must exist non-negative real numbers g_i for all $i \in I$ such that $\varphi = \sum_{i \in I} g_i \eta_i$.