MA3484 Methods of Mathematical Economics
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Example

Consider the following linear programming problem in general primal form:—

find values of x_1, x_2, x_3 and x_4 so as to minimize the objective function

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

subject to the following constraints:—

- $a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = b_1$;
- $a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 = b_2$;
- $a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + a_{3,4}x_4 \geq b_3$;
- $x_1 \geq 0$ *and* $x_3 \geq 0$.

Duality and Complementary Slackness (continued)

Here $a_{i,j}$, b_i and c_j are constants for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. The dual problem is the following:—

find values of p_1, p_2 and p_3 so as to maximize the objective function

$$p_1 b_1 + p_2 b_2 + p_3 b_3$$

subject to the following constraints:—

- $p_1 a_{1,1} + p_2 a_{2,1} + p_3 a_{3,1} \leq c_1$;
- $p_1 a_{1,2} + p_2 a_{2,2} + p_3 a_{3,2} = c_2$;
- $p_1 a_{1,3} + p_2 a_{2,3} + p_3 a_{3,3} \leq c_3$;
- $p_1 a_{1,4} + p_2 a_{2,4} + p_3 a_{3,4} = c_4$;
- $p_3 \geq 0$.

Duality and Complementary Slackness (continued)

We refer to the first and second problems as the *primal problem* and the *dual problem* respectively. Let (x_1, x_2, x_3, x_4) be a feasible solution of the primal problem, and let (p_1, p_2, p_3) be a feasible solution of the dual problem. Then

$$\begin{aligned} \sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i &= \sum_{j=1}^{4} \left(c_j - \sum_{i=1}^{3} p_i a_{i,j} \right) x_j \\ &\quad + \sum_{i=1}^{3} p_i \left(\sum_{j=1}^{4} a_{i,j} x_j - b_i \right). \end{aligned}$$

Duality and Complementary Slackness (continued)

Now the quantity $c_j - \sum_{i=1}^{3} p_i a_{i,j} = 0$ for $j = 2$ and $j = 4$, and $\sum_{j=1}^{4} a_{i,j} x_j - b_i = 0$ for $i = 1$ and $i = 2$. It follows that

$$
\begin{aligned}
\sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i &= \left(c_1 - \sum_{i=1}^{3} p_i a_{i,1} \right) x_1 \\
&\quad + \left(c_3 - \sum_{i=1}^{3} p_i a_{i,3} \right) x_3 \\
&\quad + p_3 \left(\sum_{j=1}^{4} a_{3,j} x_j - b_3 \right).
\end{aligned}
$$

Duality and Complementary Slackness (continued)

Now $x_1 \geq 0$, $x_3 \geq 0$ and $p_3 \geq 0$. Also

$$c_1 - \sum_{i=1}^{3} p_i a_{i,1} \geq 0, \quad c_3 - \sum_{i=1}^{3} p_i a_{i,3} \geq 0$$

and

$$\sum_{j=1}^{4} a_{3,j} x_j - b_3 \geq 0.$$

It follows that

$$\sum_{j=1}^{4} c_j x_j - \sum_{i=1}^{3} p_i b_i \geq 0.$$

and thus

$$\sum_{j=1}^{4} c_j x_j \geq \sum_{i=1}^{3} p_i b_i.$$

Duality and Complementary Slackness (continued)

Now suppose that

$$\sum_{j=1}^{4} c_j x_j = \sum_{i=1}^{3} p_i b_i.$$

Then

$$\begin{aligned}
\left(c_1 - \sum_{i=1}^{3} p_i a_{i,1}\right) x_1 &= 0, \\
\left(c_3 - \sum_{i=1}^{3} p_i a_{i,3}\right) x_3 &= 0, \\
p_3 \left(\sum_{j=1}^{4} a_{3,j} x_j - b_3\right) &= 0,
\end{aligned}$$

because a sum of three non-negative quantities is equal to zero if and only if each of those quantities is equal to zero.

Duality and Complementary Slackness (continued)

It follows that

$$\sum_{j=1}^{4} c_j x_j = \sum_{i=1}^{3} p_i b_i$$

if and only if the following three complementary slackness conditions are satisfied:—

- $\sum\limits_{i=1}^{3} p_i a_{i,1} = c_1$ if $x_1 > 0$;
- $\sum\limits_{i=1}^{3} p_i a_{i,3} = c_3$ if $x_3 > 0$;
- $\sum_{j=1}^{4} a_{3,j} x_j = b_3$ if $p_3 > 0$.

Open and Closed Sets in Euclidean Spaces

Let m be a positive integer. The *Euclidean norm* $|\mathbf{x}|$ of an element $\mathbf{x}$ of $\mathbb{R}^m$ is defined such that

$$|\mathbf{x}|^2 = \sum_{i=1}^{m} (\mathbf{x})_i^2.$$

Open and Closed Sets in Euclidean Spaces (continued)

The *Euclidean distance function* d on $\mathbb{R}^m$ is defined such that

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. The Euclidean distance function satisfies the Triangle Inequality, together with all the other basic properties required of a distance function on a metric space, and therefore $\mathbb{R}^m$ with the Euclidean distance function is a metric space.

Open and Closed Sets in Euclidean Spaces (continued)

A subset U of $\mathbb{R}^m$ is said to be *open* in $\mathbb{R}^m$ if, given any point $\mathbf{b}$ of U, there exists some real number ε satisfying $\varepsilon > 0$ such that

$$\{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| < \varepsilon\} \subset U.$$

A subset of $\mathbb{R}^m$ is *closed* in $\mathbb{R}^m$ if and only if its complement is open in $\mathbb{R}^m$.

Every union of open sets in $\mathbb{R}^m$ is open in $\mathbb{R}^m$, and every finite intersection of open sets in $\mathbb{R}^m$ is open in $\mathbb{R}^m$.

Every intersection of closed sets in $\mathbb{R}^m$ is closed in $\mathbb{R}^m$, and every finite union of closed sets in $\mathbb{R}^m$ is closed in $\mathbb{R}^m$.

Lemma

Lemma FK-T01 *Let m be a positive integer, let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ be a basis of $\mathbb{R}^m$, and let*

$$F = \left\{ \sum_{i=1}^{m} s_i \mathbf{u}^{(i)} : s_i \geq 0 \text{ for } i = 1, 2, \ldots, m \right\}.$$

Then F is a closed set in $\mathbb{R}^m$.

Proof

Let $T\colon \mathbb{R}^m \to \mathbb{R}^m$ be defined such that

$$T(s_1, s_2, \ldots, s_m) = \sum_{i=1}^{m} s_i \mathbf{u}^{(i)}$$

for all real numbers $s_1, s_2, \ldots, s_m$. Then T is an invertible linear operator on $\mathbb{R}^m$, and $F = T(G)$, where

$$G = \{\mathbf{x} \in \mathbb{R}^m : (\mathbf{x})_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}.$$

Moreover the subset G of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$.

Now it is a standard result of real analysis that every linear operator on a finite-dimensional vector space is continuous. Therefore $T^{-1}\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous. Moreover $T(G)$ is the preimage of the closed set G under the continuous map T^{-1}, and the preimage of any closed set under a continuous map is itself closed. It follows that $T(G)$ is closed in $\mathbb{R}^m$. Thus F is closed in $\mathbb{R}^m$, as required. ∎

Lemma

Lemma FK-CS-02 *Let m be a positive integer, let F be a non-empty closed set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.*

Proof

Let R be a positive real number chosen large enough to ensure that the set F_0 is non-empty, where

$$F_0 = F \cap \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| \leq R\}.$$

Then F_0 is a closed bounded subset of $\mathbb{R}^m$. Let $f\colon F_0 \to \mathbb{R}$ be defined such that $f(\mathbf{x}) = |\mathbf{x} - \mathbf{b}|$ for all $\mathbf{x} \in F$. Then $f\colon F_0 \to \mathbb{R}$ is a continuous function on F_0.

Now it is a standard result of real analysis that any continuous real-valued function on a closed bounded subset of a finite-dimensional Euclidean space attains a minimum value at some point of that set. It follows that there exists an element $\mathbf{g}$ of F_0 such that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F_0$. If $\mathbf{x} \in F \setminus F_0$ then

$$|\mathbf{x} - \mathbf{b}| \geq R \geq |\mathbf{g} - \mathbf{b}|.$$

It follows that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$, as required. ∎