MA3484 Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2015
Lecture 24 (March 13, 2015)

David R. Wilkins

General Linear Programming Problems, Duality and Complementary Slackness

The Nature of Linear Programming Problems

Linear programming is concerned with problems seeking to maximize or minimize a linear functional of several real variables subject to a finite collection of constraints, where each constraint either fixes the values of some linear function of the variables or else requires those values to be bounded, above or below, by some fixed quantity.

The Nature of Linear Programming Problems (continued)

The objective of such a problem involving n real variables $x_1, x_2, \ldots, x_n$ is to maximize or minimize an *objective function* of those variables that is of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

subject to appropriate constraints. The coefficients $c_1, c_2, \ldots, c_n$ that determine the objective function are then fixed real numbers.

Now such an optimization problem may be presented as a minimization problem, because simply changing the signs of all the coefficients $c_1, c_2, \ldots, c_n$ converts any maximization problem into a minimization problem. We therefore suppose, without loss of generality, that the objective of the linear programming problem is to find a feasible solution satisfying appropriate constraints which minimizes the value of the objective function amongst all such feasible solutions to the problem.

Some of the constraints may simply require specific variables to be non-negative or non-positive. Now a constraint that requires a particular variable x_j to be non-positive can be reformulated as one requiring a variable to be non-negative by substituting x_j for $-x_j$ in the statement of the problem. Thus, without loss of generality, we may suppose that all constraints that simply specify the sign of a variable x_j will require that variable to be non-negative. Then all such constraints can be specified by specifying a subset J^+ of $\{1, 2, \ldots, n\}$: the constraints then require that $x_j \geq 0$ for all $j \in J^+$.

The Nature of Linear Programming Problems (continued)

There may be further constraints in addition to those that simply specify whether one of the variables is required to be non-positive or non-negative. Suppose that there are m such additional constraints, and let them be numbered between 1 and m. Then, for each integer i between 1 and m, there exist real numbers $A_{i,1}, A_{i,2}, \ldots, A_{i,n}$ and b_i that allow the ith constraint to be expressed either as an *inequality constraint* of the form

$$A_{i,1}x_1 + A_{i,2}x_2 + \ldots + A_{i,n}x_n \geq b_i$$

or else as an *equality constraint* of the form

$$A_{i,1}x_1 + A_{i,2}x_2 + \ldots + A_{i,n}x_n = b_i.$$

The Nature of Linear Programming Problems (continued)

It follows from the previous discussion that the statement of a general linear programming problem can be transformed, by changing the signs of some of the variables and constants in the statement of the problem, so as to ensure that the statement of the problem conforms to the following restrictions:—

- the objective function is to be minimized;
- some of the variables may be required to be non-negative;
- other constraints are either inequality constraints placing a lower bound on the value of some linear function of the variables or else equality constraints fixing the value of some linear function of the variables.

The Nature of Linear Programming Problems (continued)

Let us describe the statement of a linear programming problem as being in *general primal form* if it conforms to the restrictions just described.

A linear programming problem is expressed in general primal form if the specification of the problem conforms to the following restrictions:—

- the objective of the problem is to find an optimal solution minimizing the objective function amongst all feasible solutions to the problem;
- any variables whose sign is prescribed are required to be non-negative, not non-positive;
- all inequality constraints are expressed by prescribing a lower bound on the value on some linear function of the variables.

A linear programming problem in general primal form can be specified by specifying the following data: an $m \times n$ matrix A with real coefficients, an m-dimensional vector $\mathbf{b}$ with real components; an n-dimensional vector $\mathbf{c}$ with real components; a subset I^+ of $\{1, 2, \ldots, m\}$; and a subset J^+ of $\{1, 2, \ldots, n\}$. The linear programming programming problem specified by this data is the following:—

seek $\mathbf{x} \in \mathbb{R}^n$ *that minimizes the objective function* $\mathbf{c}^T \mathbf{x}$ *subject to the following constraints:—*

- $A\mathbf{x} \geq \mathbf{b}$;
- $(A\mathbf{x})_i = (\mathbf{b})_i$ *unless* $i \in I^+$;
- $(\mathbf{x})_j \geq 0$ *for all* $j \in J^+$.

We refer to the $m \times n$ matrix A, the m-dimensional vector $\mathbf{b}$ and the n-dimensional vector $\mathbf{c}$ employed in specifying a linear programming problem in general primal form as the *constraint matrix*, *target vector* and *cost vector* respectively for the linear programming problem. Let us refer to the subset I^+ of $\{1, 2, \ldots, m\}$ specifying those constraints that are inequality constraints as the *inequality constraint specifier* for the problem, and let us refer to the subset J^+ of $\{1, 2, \ldots, n\}$ that specifies those variables that are required to be non-negative for a feasible solution as the *variable sign specifier* for the problem.

The Nature of Linear Programming Problems (continued)

We denote by $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ the linear programming problem whose specification in general primal form is determined by a constraint matrix A, target vector $\mathbf{b}$, cost vector $\mathbf{c}$, inequality constraint specifier I^+ and variable sign specifier J^+.

A linear programming problem formulated in general primal form can be reformulated as a problem in Dantzig standard form, thus enabling the use of the Simplex Method to find solutions to the problem.

Indeed consider a linear programming problem $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ where the constraint matrix A is an $m \times n$ matrix with real coefficients, the target vector $\mathbf{b}$ and the cost vector $\mathbf{c}$ are vectors of dimension m and n respectively with real coefficients. Then the inequality constraint specifier I^+ is a subset of $\{1, 2, \ldots, m\}$ and the variable sign specifier J^+ is a subset of $\{1, 2, \ldots, n\}$. The problem is already in Dantzig standard form if and only if $I^+ = \emptyset$ and $J^+ = \{1, 2, \ldots, n\}$.

The Nature of Linear Programming Problems (continued)

If the problem is not in Dantzig standard form, then each variable x_j for $j \notin J^+$ can be replaced by a pair of variables x_j^+ and x_j^- satisfying the constraints $x_j^+ \geq 0$ and $x_j^- \geq 0$: the difference $x_j^+ - x_j^-$ of these new variables is substituted for x_j in the objective function and the constraints. Also a *slack variable* z_i can be introduced for each $i \in I^+$, where z_i is required to satisfy the sign constraint $z_i \geq 0$, and the inequality constraint

$$A_{i,1}x_1 + A_{i,2}x_2 + \ldots + A_{i,n}x_n \geq b_i$$

is then replaced by the corresponding equality constraint

$$A_{i,1}x_1 + A_{i,2}x_2 + \ldots + A_{i,n}x_n - z_i = b_i.$$

The linear programming problem $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ in general primal form can therefore be reformulated as a linear programming problem in Dantzig standard form as follows:—

determine values of x_j for all $j \in J^+$, x_j^+ and x_j^- for all $j \in J^0$, where $J^0 = \{1, 2, \ldots, n\} \setminus J^+$, and z_i for all $i \in I^+$ so as to minimize the objective function

$$\sum_{j \in J^+} c_j x_j + \sum_{j \in J^0} c_j x_j^+ - \sum_{j \in J^0} c_j x_j^-$$

subject to the following constraints:—

(i) $\sum_{j \in J^+} A_{i,j} x_j + \sum_{j \in J^0} A_{i,j} x_j^+ - \sum_{j \in J^0} A_{i,j} x_j^- = b_i$ *for each* $i \in \{1, 2, \ldots, n\} \setminus I^+$;

1. $\sum_{j \in J^+} A_{i,j} x_j + \sum_{j \in J^0} A_{i,j} x_j^+ - \sum_{j \in J^0} A_{i,j} x_j^- - z_i = b_i$ *for each* $i \in I^+$;

(ii) $x_j \geq 0$ *for all* $j \in J^+$;

(iii) $x_j^+ \geq 0$ *and* $x_j^- \geq 0$ *for all* $j \in J^0$;

(iv) $z_i \geq 0$ *for all* $i \in I^+$.

Once the problem has been reformulated in Dantzig standard form, techiques based on the Simplex Method can be employed in the search for solutions to the problem.

Duality and Complementary Slackness

Every linear programming problem $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ in general primal form determines a corresponding linear programming problem $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ in *general dual form*. The second linear programming problem is referred to as the *dual* of the first, and the first linear programming problem is referred to as the *primal* of its dual.

We shall give the definition of the dual problem associated with a given linear programming problem, and investigate some important relationships between the primal linear programming problem and its dual.

Let $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ be a linear programming problem in general primal form specified in terms of an $m \times n$ constraint matrix A, m-dimensional target vector $\mathbf{b}$, n-dimensional cost vector $\mathbf{c}$, inequality constraint specifier I^+ and variable sign specifier J^+. The corresponding dual problem $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ may be specified in *general dual form* as follows:

> *seek* $\mathbf{p} \in \mathbb{R}^m$ *that maximizes the objective function* $\mathbf{p}^T\mathbf{b}$ *subject to the following constraints:—*
>
> - $\mathbf{p}^T A \leq \mathbf{c}^T$;
> - $(\mathbf{p})_i \geq 0$ *for all* $i \in I^+$;
> - $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *unless* $j \in J^+$.

Lemma

Lemma DT-01 *Let* $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ *be a linear programming problem expressed in general primal form with constraint matrix* A *with* m *rows and* n *columns, target vector* $\mathbf{b}$*, cost vector* $\mathbf{c}$*, inequality constraint specifier* I^+ *and variable sign specifier* J^+*. Then the feasible and optimal solutions of the corresponding dual linear programming problem* $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ *are those of the problem* $\mathrm{Primal}(-A^T, -\mathbf{c}, -\mathbf{b}, J^+, I^+)$*.*

Proof

An m-dimensional vector $\mathbf{p}$ satisfies the constraints of the dual linear programming problem $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ if and only if $\mathbf{p}^T A \leq \mathbf{c}^T$, $(\mathbf{p})_i \geq 0$ for all $i \in I^+$ and $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ unless $j \in J^+$. On taking the transposes of the relevant matrix equations and inequalities, we see that these conditions are satisfied if and only if $-A^T\mathbf{p} \geq -\mathbf{c}$, $(\mathbf{p})_i \geq 0$ for all $i \in I^+$ and $(-A^T\mathbf{p})_j = (-\mathbf{c})_j$ unless $j \in J^+$. But these are the requirements that the vector $\mathbf{p}$ must satisfy in order to be a feasible solution of the linear programming problem $\mathrm{Primal}(-A^T, -\mathbf{c}, -\mathbf{b}, J^+, I^+)$. Moreover $\mathbf{p}$ is an optimal solution of $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ if and only if it maximizes the value of $\mathbf{p}^T\mathbf{b}$, and this is the case if and only if it minimizes the value of $-\mathbf{b}^T\mathbf{p}$. The result follows. ∎

Duality and Complementary Slackness (continued)

A linear programming problem in Dantzig standard form is specified by specifying integers m and n a constraint matrix A which is an $m \times n$ matrix with real coefficients, a target vector $\mathbf{b}$ belonging to the real vector space $\mathbb{R}^m$ and a cost vector $\mathbf{c}$ belonging to the real vector space $\mathbb{R}^m$. The objective of the problem is to find a feasible solution to the problem that minimizes the quantity $\mathbf{c}^T\mathbf{x}$ amongst all n-dimensional vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
The objective of the dual problem is then to find some feasible solution to the problem that maximizes the quantity $\mathbf{p}^T\mathbf{b}$ amongst all m-dimensional vectors $\mathbf{p}$ for which $\mathbf{p}^T A \leq \mathbf{c}$.

Duality and Complementary Slackness (continued)

Theorem

Theorem DT-02 (Weak Duality Theorem for Linear Programming Problems in Dantzig Standard Form)
Let m and n be integers, let A be an $m \times n$ matrix with real coefficients, let $\mathbf{b} \in \mathbb{R}^m$ and let $\mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and let $\mathbf{p} \in \mathbb{R}^m$ satisfy the constraint $\mathbf{p}^T A \leq \mathbf{c}$. Then $\mathbf{p}^T \mathbf{b} \leq \mathbf{c}^T \mathbf{x}$. Moreover $\mathbf{p}^T \mathbf{b} = \mathbf{c}^T \mathbf{x}$ if and only if the following complementary slackness condition is satisfied:

- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *for all integers j between 1 and n for which $(\mathbf{x})_j > 0$.*

Proof

The constraints satisfied by the vectors $\mathbf{x}$ and $\mathbf{p}$ ensure that

$$\begin{aligned} \mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}) \\ &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x}, \end{aligned}$$

because $A\mathbf{x} - \mathbf{b} = \mathbf{0}$. But also $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^T - \mathbf{p}^T A) \geq \mathbf{0}$, and therefore $(\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} \geq 0$. Moreover $(\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} = 0$ if and only if $(\mathbf{c}^T - \mathbf{p}^T A)_j = 0$ for all integers j between 1 and n for which $(\mathbf{x})_j > 0$. The result follows. ∎

Duality and Complementary Slackness (continued)

Corollary

Corollary DT-03 *Let a linear programming problem in Dantzig standard form be specified by an* $m \times n$ *constraint matrix* A*, and* m*-dimensional target vector* $\mathbf{b}$ *and an* n*-dimensional cost vector* $\mathbf{c}$*. Let* $\mathbf{x}^*$ *be a feasible solution of this primal problem, and let* $\mathbf{p}^*$ *be a solution of the dual problem. Then* $\mathbf{p}^{*T} A \leq \mathbf{c}^T$*. Suppose that the complementary slackness conditions for this primal-dual pair are satisfied, so that* $(\mathbf{p}^{*T} A)_j = (\mathbf{c})_j$ *for all integers* j *between* 1 *and* n *for which* $(\mathbf{x}^*)_j > 0$*. Then* $\mathbf{x}^*$ *is an optimal solution of the primal problem, and* $\mathbf{p}^*$ *is an optimal solution of the dual problem.*

Proof

Because the complementary slackness conditions for this primal-dual pair are satisfied, it follows from the Weak Duality Theorem that $\mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$ (see Theorem DT-02). But it then also follows from the Weak Duality Theorem that

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^{*T}\mathbf{b} = \mathbf{c}^T\mathbf{x}^*$$

for all feasible solutions $\mathbf{x}$ of the primal problem. It follows that $\mathbf{x}^*$ is an optimal solution of the primal problem. Similarly

$$\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$$

for all feasible solutions $\mathbf{p}$ of the dual problem. It follows that $\mathbf{p}^*$ is an optimal solution of the dual problem, as required. ∎

Another special case of duality in linear programming is exemplified by a primal-dual pair of problems in *Von Neumann Symmetric Form*. In this case the primal and dual problems are specified in terms of an $m \times n$ constraint matrix A, an m-dimensional target vector $\mathbf{b}$ and an n-dimensional cost vector $\mathbf{c}$. The objective of the problem is minimize $\mathbf{c}^T\mathbf{x}$ amongst n-dimensional vectors $\mathbf{x}$ that satisfy the constraints $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. The dual problem is to maximize $\mathbf{p}^T\mathbf{b}$ amongst m-dimensional vectors $\mathbf{p}$ that satisfy the constraints $\mathbf{p}^T A \leq \mathbf{c}^T$ and $\mathbf{p} \geq \mathbf{0}$.

Duality and Complementary Slackness (continued)

Theorem

Theorem DT-04 (Weak Duality Theorem for Linear Programming Problems in Von Neumann Symmetric Form)
Let m and n be integers, let A be an $m \times n$ matrix with real coefficients, let $\mathbf{b} \in \mathbb{R}^m$ and let $\mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy the constraints $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and let $\mathbf{p} \in \mathbb{R}^m$ satisfy the constraints $\mathbf{p}^T A \leq \mathbf{c}$ and $\mathbf{p}^T \geq \mathbf{0}$. Then $\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$. Moreover $\mathbf{p}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$ if and only if the following complementary slackness conditions are satisfied:

- $(A\mathbf{x})_i = (\mathbf{b})_i$ *for all integers i between 1 and m for which* $(\mathbf{p})_i > 0$;
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *for all integers j between 1 and n for which* $(\mathbf{x})_j > 0$;

Duality and Complementary Slackness (continued)

Proof

The constraints satisfied by the vectors $\mathbf{x}$ and $\mathbf{p}$ ensure that

$$\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}).$$

But $\mathbf{x} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{0}$, $A\mathbf{x} - \mathbf{b} \geq \mathbf{0}$ and $\mathbf{c}^T - \mathbf{p}^T A \geq \mathbf{0}$. It follows that $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} \geq 0$. and therefore $\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{b}$. Moreover $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = 0$ if and only if $(\mathbf{c}^T - \mathbf{p}^T A)_j(\mathbf{x})_j = 0$ for $j = 1, 2, \ldots, n$ and $(\mathbf{p})_i(A\mathbf{x} - \mathbf{b})_i = 0$, and therefore $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$ if and only if the complementary slackness conditions are satisfied. ∎

Theorem

Theorem DT-05 (Weak Duality Theorem for Linear Programming Problems in General Primal Form)
Let $\mathbf{x} \in \mathbb{R}^{\ltimes}$ *be a feasible solution to a linear programming problem* $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ *expressed in general primal form with constraint matrix* A *with* m *rows and* n *columns, target vector* $\mathbf{b}$*, cost vector* $\mathbf{c}$*, inequality constraint specifier* I^+ *and variable sign specifier* J^+*, and let* $\mathbf{p} \in \mathbb{R}^m$ *be a feasible solution to the corresponding dual programming problem* $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$*. Then* $\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$*. Moreover* $\mathbf{p}^T\mathbf{b} = \mathbf{c}^T\mathbf{x}$ *if and only if the following complementary slackness conditions are satisfied:—*

- $(A\mathbf{x})_i = \mathbf{b}_i$ *whenever* $(\mathbf{p})_i \neq 0$*;*
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *whenever* $(\mathbf{x})_j \neq 0$*.*

Proof

The feasible solution $\mathbf{x}$ to the primal problem satisfies the following constraints:—

- $A\mathbf{x} \geq \mathbf{b}$;
- $(A\mathbf{x})_i = (\mathbf{b})_i$ unless $i \in I^+$;
- $(\mathbf{x})_j \geq 0$ for all $j \in J^+$.

The feasible solution $\mathbf{p}$ to the dual problem satisfies the following constraints:—

- $\mathbf{p}^T A \leq \mathbf{c}^T$;
- $(\mathbf{p})_i \geq 0$ for all $i \in I^+$;
- $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ unless $j \in J^+$.

Duality and Complementary Slackness (continued)

Now

$$\begin{aligned} \mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} &= (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}) \\ &= \sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j + \sum_{i=1}^{m} (\mathbf{p})_i (A\mathbf{x} - \mathbf{b})_i. \end{aligned}$$

Let j be an integer between 1 and n. If $j \in J^+$ then $(\mathbf{x})_j \geq 0$ and $(\mathbf{c}^T - \mathbf{p}^T A)_j \geq 0$, and therefore $(\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j \geq 0$. If $j \notin J^+$ then $(\mathbf{p}^T A)_j = (\mathbf{c})_j$, and therefore $(\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j = 0$, irrespective of whether $(\mathbf{x})_j$ is positive, negative or zero. It follows that

$$\sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j \geq 0.$$

Duality and Complementary Slackness (continued)

Moreover

$$\sum_{j=1}^{n} (\mathbf{c}^T - \mathbf{p}^T A)_j (\mathbf{x})_j = 0$$

if and only if $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ for all indices j for which $(\mathbf{x})_j \neq 0$.

Next let i be an index between 1 and m. If $i \in I^+$ then $(\mathbf{p})_i \geq 0$ and $(A\mathbf{x} - \mathbf{b})_i \geq 0$, and therefore $(\mathbf{p})_i (A\mathbf{x} - \mathbf{b})_i \geq 0$. If $i \notin I^+$ then $(A\mathbf{x})_i = (\mathbf{b})_i$, and therefore $(\mathbf{p})_i (A\mathbf{x} - \mathbf{b})_i = 0$, irrespective of whether $(\mathbf{p})_i$ is positive, negative or zero. It follows that

$$\sum_{i=1}^{m} (\mathbf{p})_i (A\mathbf{x} - \mathbf{p})_i \geq 0.$$

Duality and Complementary Slackness (continued)

Moreover

$$\sum_{i=1}^{m} (\mathbf{p})_i (A\mathbf{x} - \mathbf{p})_i = 0.$$

if and only if $(A\mathbf{x})_i = (\mathbf{b})_i$ for all indices i for which $(\mathbf{p})_i \neq 0$. The result follows. $\blacksquare$

Duality and Complementary Slackness (continued)

Corollary

Corollary DT-06 *Let* $\mathbf{x}^* \in \mathbb{R}^{\ltimes}$ *be a feasible solution to a linear programming problem* $\mathrm{Primal}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$ *expressed in general primal form with constraint matrix* A *with* m *rows and* n *columns, target vector* $\mathbf{b}$, *cost vector* $\mathbf{c}$, *inequality constraint specifier* I^+ *and variable sign specifier* J^+, *and let* $\mathbf{p}^* \in \mathbb{R}^m$ *be a feasible solution to the corresponding dual programming problem* $\mathrm{Dual}(A, \mathbf{b}, \mathbf{c}, I^+, J^+)$. *Suppose that the complementary slackness conditions are satisfied for this pair of problems, so that* $(A\mathbf{x})_i = \mathbf{b}_i$ *whenever* $(\mathbf{p})_i \neq 0$, *and* $(\mathbf{p}^T A)_j = (\mathbf{c})_j$ *whenever* $(\mathbf{x})_j \neq 0$. *Then* $\mathbf{x}^*$ *is an optimal solution for the primal problem and* $\mathbf{p}^*$ *is an optimal solution for the dual problem.*

Proof

Because the complementary slackness conditions for this primal-dual pair are satisfied, it follows from the Weak Duality Theorem that $\mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$ (see Theorem DT-05). But it then also follows from the Weak Duality Theorem that

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^{*T}\mathbf{b} = \mathbf{c}^T\mathbf{x}^*$$

for all feasible solutions $\mathbf{x}$ of the primal problem. It follows that $\mathbf{x}^*$ is an optimal solution of the primal problem. Similarly

$$\mathbf{p}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}^* = \mathbf{p}^{*T}\mathbf{b}$$

for all feasible solutions $\mathbf{p}$ of the dual problem. It follows that $\mathbf{p}^*$ is an optimal solution of the dual problem, as required. ∎