MA3484 Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2015
Lecture 22 (March 11, 2015)

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The Extended Simplex Tableau for solving Linear Programming Problems

We now consider the construction of a tableau for a linear programming problem in Dantzig standard form.

We summarize the construction of the extended simplex tableau for a linear programming problem in Dantzig standard form. A more detailed account was given in the previous lecture.

The Extended Simplex Tableau (continued)

Such a problem is specified by an $m \times n$ matrix A, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. We suppose moreover that the matrix A is of rank m. We consider procedures for solving the following linear program in Danzig standard form.

> *Determine* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to the constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

A *feasible* solution to this problem consists of an n-dimensional vector $\mathbf{x}$ that satisfies the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. An *optimal solution* to the problem is a feasible solution that minimizes the *objective function* $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the problem.

The Extended Simplex Tableau (continued)

We denote by $\mathbf{a}^{(j)}$ the m-dimensional column vector whose components are those in the jth column of the matrix A. A feasible solution of the linear programming problem then consists of non-negative real numbers $x_1, x_2, \ldots, x_n$ for which

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \mathbf{b}.$$

A feasible solution determined by $x_1, x_2, \ldots, x_n$ is optimal if it minimizes cost $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the linear programming problem.

The Extended Simplex Tableau (continued)

A *basis* B for the linear programming problem is a subset of $\{1, 2, \ldots, n\}$ with m elements which has the property that the vectors $\mathbf{a}^{(j)}$ for $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$.

In the following discussion we let

$$B = \{j_1, j_2, \ldots, j_m\},$$

where $j_1, j_2, \ldots, j_m$ are distinct integers between 1 and n.

We denote by M_B the invertible $m \times m$ matrix whose component $(M)_{i,k}$ in the ith row and jth column satisfies $(M_B)_{i,k} = (A)_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then the kth column of the matrix M_B is specified by the column vector $\mathbf{a}^{(j_k)}$ for $k = 1, 2, \ldots, m$, and thus the columns of the matrix M_B coincide with those columns of the matrix A that are determined by elements of the basis B.

The Extended Simplex Tableau (continued)

Every vector in $\mathbb{R}^m$ can be expressed as a linear combination of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$. It follows that there exist uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

It follows from Lemma STG-01 that

$$t_{i,j} = (M_B^{-1} \mathbf{a}^{(j)})_i \quad \text{and} \quad s_i = (M_B^{-1} \mathbf{b})_i$$

for $i = 1, 2, \ldots, m$.

The Extended Simplex Tableau (continued)

The standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$ is defined such that

$$(\mathbf{e}^{(k)})_i = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

It follows from Lemma STG-02 that

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k} \mathbf{u}^{(i)},$$

where $r_{i,k}$ is the coefficient $(M_B^{-1})_{i,k}$ in the ith row and kth column of the inverse M_B^{-1} of the matrix M_B.

The Extended Simplex Tableau (continued)

The *cost* C of the basic feasible solution $\mathbf{x}$ is defined to be the value $\overline{c}^T\mathbf{x}$ of the objective function. The definition of the quantities $s_1, s_2, \ldots, s_m$ ensures that

$$C = \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} c_{j_i} s_i.$$

If the quantities $s_1, s_2, \ldots, s_n$ are not all non-negative then there is no basic feasible solution associated with the basis B.

The Extended Simplex Tableau (continued)

The extended simplex tableau has cells, located in the criterion row at the bottom of the tableau to record quantities $p_1, p_2, \ldots, p_m$ associated with the vectors that constitute the standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$. These quantities are defined so that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$, where c_{j_i} is the cost associated with the basis vector $\mathbf{a}^{(j_i)}$ for $i = 1, 2, \ldots, k$,

The Extended Simplex Tableau (continued)

An application of Lemma STG-03 establishes that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = c_{j_i}$$

for $i = 1, 2, \ldots, k$.

When this identity is combined with previous identities, we find that

$$C = \sum_{i=1}^{m} c_{j_i} s_i = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{j_i} r_{i,k} b_k = \sum_{k=1}^{m} p_k b_k.$$

The Extended Simplex Tableau (continued)

The extended simplex tableau also has cells in the criterion row to record quantities $-q_1, -q_2, \ldots, -q_n$ that satisfy the identities

$$q_{j_i} = c_{j_i} - (\mathbf{p}^T A)_{j_i} = c_{j_i} - \sum_{k=1}^{m} p_k A_{k,j_i} = 0$$

for $i = 1, 2, \ldots, n$.

The extended simplex tableau records the values of the quantities $t_{i,j}$, s_i, $r_{i,k}$, p_k, C and q_i for all integers i and k between 1 and m and for all integers j between 1 and n in a tableau with the following structure:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C	p_1	p_2	$\cdots$	p_m

Simplex Tableau Example (continued)

The extended simplex tableau can be represented in block form as follows:—

	$\mathbf{a}^{(1)}$ ... $\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$ ... $\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ ⋮ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

Simplex Tableau Example (continued)

Let $\mathbf{c}_B$ denote the m-dimensional vector defined so that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{pmatrix}.$$

The identities we have verified ensure that the extended simplex tableau can therefore also be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1}A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$

The Extended Simplex Tableau (continued)

Given an $m \times n$ matrix A of rank m, an m-dimensional target vector $\mathbf{b}$, and an n-dimensional cost vector $\mathbf{c}$, there exists an extended simplex tableau associated with any basis B for the linear programming problem, irrespective of whether or not there exists a basic feasible solution associated with the given basis B.

The crucial requirement that enables the construction of the tableau is that the basis B should consist of m distinct integers $j_1, j_2, \ldots, j_m$ between 1 and m for which the corresponding columns of the matrix A constitute a basis of the vector space $\mathbb{R}^m$.

The Extended Simplex Tableau (continued)

A basis B is associated with a basic feasible solution of the linear programming problem if and only if the values in the column labelled by the target vector $\mathbf{b}$ and the rows labelled by $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ should be non-negative. If so, those values will include the non-zero components of the basic feasible solution associated with the basis.

If there exists a basic feasible solution associated with the basis B then that solution is optimal if and only if all the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ are all non-positive.

Versions of the Simplex Tableau Algorithm for determining a basic optimal solution to the linear programmming problem, given an initial basic feasible solution, rely on the transformation rules that determine how the values in the body of the extended simplex tableau are transformed on passing from an old basis B to an new basis B', where the new basis B' contains all but one of the members of the old basis B.

The Extended Simplex Tableau (continued)

Let us refer to the rows of the extended simplex tableau labelled by the basis vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ as the *basis rows* of the tableau.

Lemma STG-04 determines how entries in the basis rows of the extended simplex tableau transform which one element of the basis is replaced by an element not belonging to the basis.

Let the old basis B consist of distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n, and let the new basis B' also consist of distinct integers $j'_1, j'_2, \ldots, j'_m$ between 1 and n. We suppose that the new basis B' is obtained from the old basis by replacing an element j_h of the old basis B by some integer j'_h between 1 and n that does not belong to the old basis. We suppose therefore that $j_i = j'_i$ when $i \neq h$, and that j'_h is some integer between 1 and n that does not belong to the basis B.

The Extended Simplex Tableau (continued)

Let the coefficients $t_{i,j}$, $t'_{i,j}$, s_i, s'_i, $r_{i,k}$ and $r'_{i,k}$ be determined for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j}\mathbf{a}^{(j'_i)}$$

for $j = 1, 2, \ldots, n$,

$$\mathbf{b} = \sum_{i=1}^{m} s_i\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s'_i\mathbf{a}^{(j'_i)}$$

and

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} r'_{i,k}\mathbf{a}^{(j'_i)}$$

for $k = 1, 2, \ldots, m$.

The Extended Simplex Tableau (continued)

It then follows from Lemma STG-04 that

$$
\begin{aligned}
t'_{h,j} &= \frac{1}{t_{h,j'_h}}\, t_{h,j}, \\
t'_{i,j} &= t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, t_{h,j} \quad (i \neq h). \\
s'_h &= \frac{1}{t_{h,j'_h}}\, s_h, \\
s'_i &= s_i - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, s_h \quad (i \neq h),
\end{aligned}
$$

Simplex Tableau Example (continued)

$$\begin{aligned} r'_{h,k} &= \frac{1}{t_{h,j'_h}}\, r_{h,k}, \\ r'_{i,k} &= r_{i,k} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, r_{h,k} \quad (i \neq h). \end{aligned}$$

The Extended Simplex Tableau (continued)

The *pivot row* of the extended simplex tableau for this change of basis from B to B' is the row labelled by the basis vector $\mathbf{a}^{(j_h)}$ that is to be removed from the current basis. The *pivot column* of the extended simplex tableau for this change of basis is the column labelled by the vector $\mathbf{a}^{(j'_h)}$ that is to be added to the current basis. The *pivot element* for this change of basis is the element t_{h,j'_h} of the tableau located in the pivot row and pivot column of the tableau.

The Extended Simplex Tableau (continued)

The identities relating the components of $\mathbf{a}^{(j)}$, $\mathbf{b}$ and $\mathbf{e}^{(k)}$ with respect to the old basis to the components of those vectors with respect to the new basis ensure that the rules for transforming the rows of the tableau other than the criterion row can be stated as follows:—

- a value recorded in the pivot row is transformed by dividing it by the pivot element;
- an value recorded in a basis row other than the pivot row is transformed by substracting from it a constant multiple of the value in the same column that is located in the pivot row, where this constant multiple is the ratio of the values in the basis row and pivot row located in the pivot column.

The Extended Simplex Tableau (continued)

In order to complete the discussion of the rules for transforming the values recorded in the extended simplex tableau under a change of basis that replaces an element of the old basis by an element not in that basis, it remains to analyse the rule that determines how the elements of the criterion row are transformed under this change of basis.

Simplex Tableau Example (continued)

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, \ldots, n$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j}\mathbf{a}^{(j'_i)},$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$. Moreover

$$t'_{h,j} = \frac{1}{t_{h,j'_h}}\, t_{h,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}}\, t_{h,j}$$

for all integers i between 1 and m for which $i \neq h$.

Simplex Tableau Example (continued)

Now

$$-q_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j$$

and

$$-q'_j = \sum_{i=1}^{m} c_{j'_i} t'_{i,j} - c_j.$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} q_j - q'_j &= \sum_{\substack{1 \le i \le m \\ i \neq h}} c_{j_i}(t'_{i,j} - t_{i,j}) + c_{j'_h} t'_{h,j} - c_{j_h} t_{h,j} \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) t_{h,j} \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} \, t_{h,j} \end{aligned}$$

and thus

$$-q'_j = -q_j + \frac{q_{j'_h}}{t_{h,j'_h}} \, t_{h,j}$$

for $j = 1, 2, \ldots, n$.

Simplex Tableau Example (continued)

Next we note that

$$C = \sum_{i=1}^{m} c_{j_i} s_i$$

and

$$C' = \sum_{i=1}^{m} c_{j'_i} s'_i .$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} C' - C &= \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_{j_i}(s'_i - s_i) + c_{j'_h} s'_h - c_{j_h} s_h \\ &= \frac{1}{t_{h,j'_h}} \left(- \sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) s_h \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} s_h \end{aligned}$$

and thus

$$C' = q_k + \frac{q_{j'_h}}{t_{h,j'_h}} s_h$$

for $k = 1, 2, \ldots, m$.

Simplex Tableau Example (continued)

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

and

$$p'_k = \sum_{i=1}^{m} c_{j'_i} r'_{i,k}.$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} p'_k - p_k &= \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_{j_i}(r'_{i,k} - r_{i,k}) + c_{j'_h} r'_{h,k} - c_{j_h} r_{h,k} \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) r_{h,k} \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k} \end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k}$$

for $k = 1, 2, \ldots, m$.

We conclude that the criterion row of the extended simplex tableau transforms under changes of basis that replace one element of the basis according to a rule analogous to that which applies to the basis rows. Indeed an element of the criterion row is transformed by subtracting from it a constant multiple of the element in the pivot row that belongs to the same column, where the multiplying factor is the ratio of the elements in the criterion row and pivot row of the pivot column.

The Extended Simplex Tableau (continued)

We have now discussed how the extended simplex tableau associated with a given basis B is constructed from the constraint matrix A, target vector $\mathbf{b}$ and cost vector $\mathbf{c}$ that characterizes the linear programming problem. We have also discussed how the tableau transforms when one element of the given basis is replaced.

It remains how to replace an element of a basis associated with a non-optimal feasible solution so as to obtain a basic feasible solution of lower cost where this is possible.

The Extended Simplex Tableau (continued)

We use the notation previously established. Let $j_1, j_2, \ldots, j_m$ be the elements of a basis B that is associated with some basic feasible solution of the linear programming problem. Then there are non-negative numbers $s_1, s_2, \ldots, s_m$ such that

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)},$$

where $\mathbf{a}^{(j_i)}$ is the m-dimensional vector determined by column j_i of the constraint matrix A.

The Extended Simplex Tableau (continued)

Let j_0 be an integer between 1 and n that does not belong to the basis B. Then

$$\mathbf{a}^{(j_0)} - \sum_{i=1}^{m} t_{i,j_0} \mathbf{a}^{(j_i)} = \mathbf{0}.$$

and therefore

$$\lambda \mathbf{a}^{(j_0)} + \sum_{i=1}^{m} (s_i - \lambda t_{i,j_0}) \mathbf{a}^{(j_i)} = \mathbf{b}.$$

This expression representing $\mathbf{b}$ as a linear combination of vectors $\mathbf{a}^{(j_0)}, \mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ determines an n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ satisfying the matrix equation $A\overline{\mathbf{x}}(\lambda) = \mathbf{b}$. Let $\overline{x}_j(\lambda)$ denote the jth component of the vector $\overline{\mathbf{x}}(\lambda)$ for $j = 1, 2, \ldots, n$. Then

- $\overline{x}_{j_0}(\lambda) = \lambda$;
- $\overline{x}_{j_i}(\lambda) = s_i - \lambda t_{i,j_0}$ for $i = 1, 2, \ldots, m$;
- $\overline{x}_j = 0$ when $j \notin \{j_0, j_1, j_2, \ldots, j_m\}$.

The n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ represents a feasible solution of the linear programming problem if and only if all its coefficients are non-negative. The cost is then $C + q_{j_0}\lambda$, where C is the cost of the basic feasible solution determined by the basis B.

Suppose that $q_{j_0} < 0$ and that $t_{i,j_0} \leq 0$ for $i = 1, 2, \ldots, m$. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution with cost $C + q_{j_0}\lambda$ for all non-negative real numbers λ. In this situation there is no optimal solution to the linear programming problem, because, given any real number K, it is possible to choose λ so that $C + q_{j_0}\lambda < K$, thereby obtaining a feasible solution whose cost is less than K.

The Extended Simplex Tableau (continued)

If there does exist an optimal solution to the linear programming problem then there must exist at least one integer i between 1 and m for which $t_{i,j_0} > 0$. We suppose that this is the case. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution if and only if λ satisfies $0 \leq \lambda \leq \lambda_0$, where

$$\lambda_0 = \mathrm{minimum}\left(\frac{s_i}{t_{i,j_0}} : t_{i,j_0} > 0\right).$$

The Extended Simplex Tableau (continued)

We can then choose some integer h between 1 and n for which

$$\frac{s_h}{t_{h,j_0}} = \lambda_0.$$

Let $j'_i = j_i$ for $i \neq h$, and let $j'_h = j_0$, and let $B' = \{j'_1, j'_2, \ldots, j'_m\}$. Then $\overline{\mathbf{x}}(\lambda_0)$ is a basic feasible solution of the linear programming problem associated with the basis B'. The cost of this basic feasible solution is

$$C + \frac{s_h q_{j_0}}{t_{h,j_0}}.$$

It makes sense to select the replacement column so as to obtain the greatest cost reduction. The procedure for finding this information from the tableau can be described as follows.

The Extended Simplex Tableau (continued)

We suppose that the simplex tableau for a basic feasible solution has been prepared. Examine the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$. If all those are non-positive then the basic feasible solution is optimal. If not, then consider in turn those columns $\mathbf{a}^{(j_0)}$ for which the value $-q_{j_0}$ in the criterion row is positive. For each of these columns, examine the coefficients recorded in the column in the basis rows. If these coefficients are all non-positive then there is no optimal solution to the linear programming problem. Otherwise choose h to be the value of i that minimizes the ratio $\dfrac{s_i}{t_{i,j_0}}$ amongst those values of i for which $t_{i,j_0} > 0$. The row labelled by $\mathbf{a}^{(j_h)}$ would then be the pivot row if the column $\mathbf{a}^{(j_0)}$ were used as the pivot column.

The Extended Simplex Tableau (continued)

Calculate the value of the cost reduction $\dfrac{s_h(-q_{j_0})}{t_{h,j_0}}$ that would result if the column labelled by $\mathbf{a}^{(j_0)}$ were used as the pivot column. Then choose the pivot column to maximize the cost reduction amongst all columns $\mathbf{a}^{(j_0)}$ for which $-q_{j_0} > 0$. Choose the row labelled by $\mathbf{a}^{(j_h)}$, where h is determined as described above. Then apply the procedures for transforming the simplex tableau to that determined by the new basis B', where B' includes j_0 together with j_i for all integers i between 1 and m satisfying $i \neq h$.