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David R. Wilkins

We now calculate the new values for the criterion row. The new basis B' is given by $B' = \{j'_1, j'_2, j'_3\}$, where $j'_1 = 1$, $j'_2 = 4$ and $j'_3 = 3$. The values p'_1, p'_2 and p'_3 that are to be recorded in the criterion row of the new tableau in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ respectively are determined by the equation

$$p'_k = c_{j'_1} r'_{1,k} + c_{j'_2} r'_{2,k} + c_{j'_3} r'_{3,k}$$

for $k = 1, 2, 3$, where

$$c_{j'_1} = c_1 = 2, \quad c_{j'_2} = c_4 = 1, \quad c_{j'_3} = c_3 = 3,$$

and where $r'_{i,k}$ denotes the ith component of the vector $\mathbf{e}^{(k)}$ with respect to the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(4)}, \mathbf{a}^{(3)}$ of $\mathbb{R}^3$.

Simplex Tableau Example (continued)

We find that

$$
\begin{aligned}
p'_1 &= c_{j'_1} r'_{1,1} + c_{j'_2} r'_{2,1} + c_{j'_3} r'_{3,1} \\
&= 2 \times \left(-\tfrac{9}{27}\right) + 1 \times \tfrac{6}{27} + 3 \times \tfrac{6}{27} = \tfrac{6}{27}, \\
p'_2 &= c_{j'_1} r'_{1,2} + c_{j'_2} r'_{2,2} + c_{j'_3} r'_{3,2} \\
&= 2 \times \tfrac{12}{27} + 1 \times \tfrac{7}{27} + 3 \times \left(-\tfrac{11}{27}\right) = -\tfrac{2}{27}, \\
p'_3 &= c_{j'_1} r'_{1,3} + c_{j'_2} r'_{2,3} + c_{j'_3} r'_{3,3} \\
&= 2 \times \tfrac{3}{27} + 1 \times \left(-\tfrac{5}{27}\right) + 3 \times \tfrac{4}{27} = \tfrac{13}{27}.
\end{aligned}
$$

We next calculate the cost C' of the new basic feasible solution. The quantities s'_1, s'_2 and s'_3 satisfy $s'_i = x'_{j_i}$ for $i = 1, 2, 3$, where $(x'_1, x'_2, x'_3, x'_4, x'_5)$ is the new basic feasible solution. It follows that

$$C' = c_{j_1} s'_1 + c_{j_2} s'_2 + c_{j_3} s'_3,$$

where s_1, s_2 and s_3 are determined so that

$$\mathbf{b} = s'_1 \mathbf{a}^{(j'_1)} + s'_2 \mathbf{a}^{(j'_2)} + s'_3 \mathbf{a}^{(j'_3)}.$$

The values of s'_1, s'_2 and s'_3 have already been determined, and have been recorded in the column of the new tableau labelled by the vector $\mathbf{b}$.

Simplex Tableau Example (continued)

We can therefore calculate C' as follows:—

$$\begin{aligned} C' &= c_{j'_1} s'_1 + c_{j_2} s'_2 + c_{j_3} s'_3 = c_1 s'_1 + c_4 s'_2 + c_3 s'_3 \\ &= 2 \times \tfrac{99}{27} + \tfrac{69}{27} + 3 \times \tfrac{15}{27} = \tfrac{312}{27}. \end{aligned}$$

Alternatively we can use the identity $C' = \mathbf{p}'^T \mathbf{b}$ to calculate C' as follows:

$$C' = p'_1 b_1 + p'_2 b_2 + p'_3 b_3 = \tfrac{6}{27} \times 13 - \tfrac{2}{27} \times 13 + \tfrac{13}{27} \times 20 = \tfrac{312}{27}.$$

Simplex Tableau Example (continued)

We now enter the values of p_1', p_2', p_3' and C' into the tableau associated with basis $\{1, 4, 3\}$. The tableau then takes the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	·	·	·	·	·	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

Simplex Tableau Example (continued)

In order to complete the extended tableau, it remains to calculate the values $-q'_j$ for $j = 1, 2, 3, 4, 5$, where q'_j satisfies the equation $-q'_j = \mathbf{p}'^T \mathbf{a}_j - c_j$ for $j = 1, 2, 3, 4, 5$.

Now q'_j is the jth component of the vector $\mathbf{q}'$ that satisfies the matrix equation $-\mathbf{q}'^T = \mathbf{p}'^T A - \mathbf{c}^T$. It follows that

$$\begin{aligned} -\mathbf{q}'^T &= \mathbf{p}'^T A - \mathbf{c}^T \\ &= \begin{pmatrix} \frac{6}{27} & \frac{-2}{27} & \frac{13}{27} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix} \\ &\qquad - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & \frac{32}{27} & 3 & 1 & \frac{76}{27} \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{76}{27} & 0 & 0 & -\frac{32}{27} \end{pmatrix} \end{aligned}$$

Simplex Tableau Example (continued)

Thus

$$q'_1 = 0, \quad q'_2 = \tfrac{76}{27}, \quad q'_3 = 0, \quad q'_4 = 0, \quad q'_5 = \tfrac{32}{27}.$$

The value of each q'_j can also be calculated from the other values recorded in the column of the extended simplex tableau labelled by the vector $\mathbf{a}^{(j)}$. Indeed the vector $\mathbf{p}'$ is determined so as to satisfy the equation $\mathbf{p}'^T \mathbf{a}^{(j')} = c_{j'}$ for all $j' \in B'$. It follows that

$$\mathbf{p}'^T \mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j} \mathbf{p}'^T \mathbf{a}^{(j'_i)} = \sum_{i=1}^{3} c_{j'_i} t'_{i,j},$$

and therefore

$$-q'_j = \sum_{i=1}^{3} c_{j'_i} t'_{i,j} - c_j.$$

Simplex Tableau Example (continued)

The extended simplex tableau for the basis $\{1, 4, 3\}$ has now been computed, and the completed tableau is as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

Simplex Tableau Example (continued)

The fact that $q'_j \geq 0$ for $j = 1, 2, 3, 4, 5$ shows that we have now found our basic optimal solution. Indeed the cost $\overline{C}$ of any feasible solution $\overline{\mathbf{x}}$ satisfies

$$\begin{aligned} \overline{C} &= \mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}'^T A\overline{\mathbf{x}} + \mathbf{q}'^T\overline{\mathbf{x}} = \mathbf{p}'^T\mathbf{b} + \mathbf{q}'^T\overline{\mathbf{x}} \\ &= C' + \mathbf{q}'^T\overline{\mathbf{x}} \\ &= C' + \frac{76}{27}\overline{x}_2 + \frac{32}{27}\overline{x}_5, \end{aligned}$$

where $\overline{x}_2 = (\overline{\mathbf{x}})_2$ and $\overline{x}_5 = (\overline{\mathbf{x}})_5$.
Therefore $\mathbf{x}'$ is a basic optimal solution to the linear programming problem, where

$$\mathbf{x}'^T = \left(\begin{array}{ccccc} \frac{99}{27} & 0 & \frac{15}{27} & \frac{69}{27} & 0 \end{array} \right).$$

Simplex Tableau Example (continued)

It is instructive to compare the pivot row and criterion row of the tableau for the basis $\{1, 2, 3\}$ with the corresponding rows of the tableau for the basis $\{1, 4, 3\}$.

Simplex Tableau Example (continued)

These rows in the old tableau for the basis $\{1, 2, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

The corresponding rows in the new tableau for the basis $\{1, 4, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{23}$	$\frac{13}{23}$

Simplex Tableau Example (continued)

If we examine the values of the criterion row in the new tableau we find that they are obtained from corresponding values in the criterion row of the old tableau by subtracting off the corresponding elements of the pivot row of the old tableau multiplied by the factor $\frac{76}{27}$. As a result, the new tableau has value 0 in the cell of the criterion row in column $\mathbf{a}^{(4)}$. Thus the same rule used to calculate values in other rows of the new tableau would also have yielded the correct elements in the criterion row of the tableau.

We now investigate the reasons why this is so.

Simplex Tableau Example (continued)

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, 3, 4, 5$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} t'_{i,j}\mathbf{a}^{(j'_i)},$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$. Moreover

$$t'_{2,j} = \frac{1}{t_{2,4}}\, t_{2,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, t_{2,j} \quad (i = 1, 3).$$

Simplex Tableau Example (continued)

Now

$$\begin{aligned} -q_j &= \sum_{i=1}^{3} c_{j_i} t_{i,j} - c_j \\ &= c_1 t_{1,j} + c_2 t_{2,j} + c_3 t_{3,j} - c_j, \\ -q'_j &= \sum_{i=1}^{3} c_{j'_i} t'_{i,j} - c_j. \\ &= c_1 t'_{1,j} + c_4 t'_{2,j} + c_3 t'_{3,j} - c_j. \end{aligned}$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} q_j - q'_j &= c_1(t'_{1,j} - t_{1,j}) + c_4 t'_{2,j} - c_2 t_{2,j} + c_3(t'_{3,j} - t_{3,j}) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) t_{2,j} \\ &= \frac{q_4}{t_{2,4}} t_{2,j} \end{aligned}$$

and thus

$$-q'_j = -q_j + \frac{q_4}{t_{2,4}} t_{2,j}$$

for $j = 1, 2, 3, 4, 5$.

Simplex Tableau Example (continued)

Next we note that

$$
\begin{aligned}
C &= \sum_{i=1}^{3} c_{j_i} s_i = c_1 s_1 + c_2 s_2 + c_3 s_3, \\
C' &= \sum_{i=1}^{3} c_{j_i'} s_i' = c_1 s_1' + c_4 s_2' + c_3 s_3'.
\end{aligned}
$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} C' - C &= c_1(s_1' - s_1) + c_4 s_2' - c_2 s_2 + c_3(s_3' - s_3) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) s_2 \\ &= \frac{q_4}{t_{2,4}} s_2 \end{aligned}$$

and thus

$$C' = q_k + \frac{q_4}{t_{2,4}} s_2$$

for $k = 1, 2, 3$.

Simplex Tableau Example (continued)

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$\begin{aligned} p_k &= \sum_{i=1}^{3} c_{j_i} r_{i,k} = c_1 r_{1,k} + c_2 r_{2,k} + c_3 r_{3,k}, \\ p'_k &= \sum_{i=1}^{3} c_{j'_i} r'_{i,k} = c_1 r'_{1,k} + c_4 r'_{2,k} + c_3 r'_{3,k}. \end{aligned}$$

Simplex Tableau Example (continued)

Therefore

$$\begin{aligned} p'_k - p_k &= c_1(r'_{1,k} - r_{1,k}) + c_4 r'_{2,k} - c_2 r_{2,k} + c_3(r'_{3,k} - r_{3,k}) \\ &= \frac{1}{t_{2,4}} \left(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4}\right) r_{2,k} \\ &= \frac{q_4}{t_{2,4}} \, r_{2,k} \end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_4}{t_{2,4}} \, r_{2,k}$$

for $k = 1, 2, 3$.

This completes the discussion of the structure and properties of the extended simplex tableau associated with the optimization problem under discussion.

Some Results concerning Finite-Dimensional Real Vector Spaces

We consider the representation of vectors belonging to the m-dimensional vector space $\mathbb{R}^m$ as linear combinations of basis vectors belonging to some chosen basis of this m-dimensional real vector space.

Elements of $\mathbb{R}^m$ are normally considered to be *column vectors* represented by $m \times 1$ matrices. Given any $\mathbf{v} \in \mathbb{R}^m$, we denote by $(\mathbf{v})_i$ the ith component of the vector $\mathbf{v}$, and we denote by $\mathbf{v}^T$ the $1 \times m$ row vector that is the transpose of the column vector representing $\mathbf{v} \in \mathbb{R}^m$. Thus

$$\mathbf{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix},$$

where $v_i = (\mathbf{v})_i$ for $i = 1, 2, \ldots, m$.

Linear Algebra Results (continued)

We define the *standard basis* of the real vector space $\mathbb{R}^m$ to be the basis

$$\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$$

defined such that

$$(\mathbf{e}^{(k)})_i = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

It follows that $\mathbf{v} = \sum_{i=1}^{m} (\mathbf{v})_i \mathbf{e}^{(i)}$ for all $\mathbf{v} \in \mathbb{R}^m$.

Let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ be a basis of the real vector space $\mathbb{R}^m$, and let $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ denote the standard basis of $\mathbb{R}^m$. Then there exists an invertible $m \times m$ matrix M with the property that

$$\mathbf{u}^{(k)} = \sum_{i=1}^{m} (M)_{i,k} \mathbf{e}^{(i)}$$

for $k = 1, 2, \ldots, m$.

Linear Algebra Results (continued)

The product $M\mathbf{v}$ is defined in the usual fashion for any m-dimensional vector $\mathbf{v}$: the vector $\mathbf{v}$ is expressed as an $m \times 1$ column vector, and the matrix product is then calculated according to the usual rules of matrix multiplication, so that

$$(M\mathbf{v})_i = \sum_{k=1}^{m} (M)_{i,k} (\mathbf{v})_k,$$

and thus

$$M\mathbf{v} = \sum_{i=1}^{m} \sum_{k=1}^{m} M_{i,k} (\mathbf{v})_k \mathbf{e}^{(i)} = \sum_{k=1}^{m} (\mathbf{v})_k \mathbf{u}^{(k)}.$$

Then $M\mathbf{e}^{(i)} = \mathbf{u}^{(i)}$ for $i = 1, 2, \ldots, m$. The inverse matrix M^{-1} of M then satisfies $M^{-1}\mathbf{u}^{(i)} = \mathbf{e}^{(i)}$ for $i = 1, 2, \ldots, m$.

Lemma

Lemma STG-01 *Let m be a positive integer, let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ be a basis of $\mathbb{R}^m$, let $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ denote the standard basis of $\mathbb{R}^m$, and let M be the non-singular matrix that satisfies $M\mathbf{e}^{(i)} = \mathbf{u}^{(i)}$ for $i = 1, 2, \ldots, m$. Let $\mathbf{v}$ be a vector in $\mathbb{R}^m$, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the unique real numbers for which*

$$\mathbf{v} = \sum_{i=1}^{m} \lambda_i \mathbf{u}^{(i)}.$$

Then λ_i is the ith component of the vector $M^{-1}\mathbf{v}$ for $i = 1, 2, \ldots, m$.

Proof

The inverse matrix M^{-1} of M satisfies $M^{-1}\mathbf{u}^{(k)} = \mathbf{e}^{(k)}$ for $k = 1, 2, \ldots, m$. It follows that

$$M^{-1}\mathbf{v} = \sum_{k=1}^{m} \lambda_k M^{-1}\mathbf{u}^{(k)} = \sum_{k=1}^{m} \lambda_k \mathbf{e}^{(k)},$$

and thus $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the components of the column vector $M^{-1}\mathbf{v}$, as required. ∎

Lemma

Lemma STG-02 *Let m be a positive integer, let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ be a basis of $\mathbb{R}^m$, let $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ denote the standard basis of $\mathbb{R}^m$, and let M be the non-singular matrix that satisfies $M\mathbf{e}^{(i)} = \mathbf{u}^{(i)}$ for $i = 1, 2, \ldots, m$. Then*

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k}\mathbf{u}^{(i)},$$

where $r_{i,k}$ is the coefficient $(M^{-1})_{i,k}$ in the ith row and kth column of the inverse M^{-1} of the matrix M.

Proof

It follows from Lemma STG-01 that $\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k}\mathbf{u}^{(i)}$, where $r_{i,k} = (M^{-1}\mathbf{e}^{(k)})_i$ for $i = 1, 2, \ldots, m$. But $M^{-1}\mathbf{e}^{(k)}$ is the column vector whose components are those of the kth column of the matrix M^{-1}. The result follows. $\blacksquare$

Lemma

Lemma STG-03 *Let* m *be a positive integer, let* $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ *be a basis of* $\mathbb{R}^m$*, let* $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ *denote the standard basis of* $\mathbb{R}^m$*, let* M *be the non-singular matrix that satisfies* $M\mathbf{e}^{(i)} = \mathbf{u}^{(i)}$ *for* $i = 1, 2, \ldots, m$*, and let* $r_{i,k} = (M^{-1})_{i,k}$ *for* $i = 1, 2, \ldots, m$ *and* $k = 1, 2, \ldots, m$*. Let* $g_1, g_2, \ldots, g_m$ *be real numbers, and let* $\mathbf{p} = \sum_{k=1}^{m} p_k \mathbf{e}^{(k)}$*, where* $p_k = \sum_{i=1}^{m} g_i r_{i,k}$ *for* $k = 1, 2, \ldots, m$*. Then* $\mathbf{p}^T \mathbf{u}^{(i)} = g_i$ *for* $i = 1, 2, \ldots, m$*.*

Proof

It follows from the definition of the matrix M that $(\mathbf{u}^{(i)})_k = (M)_{k,i}$ for all integers i and k between 1 and m. It follows that the ith component of the row vector $\mathbf{p}^T M$ is equal to $\mathbf{p}^T \mathbf{u}^{(i)}$ for $i = 1, 2, \ldots, m$. But the definition of the vector $\mathbf{p}$ ensures that p_i is the ith component of the row vector $\mathbf{g}^T M^{-1}$, where $\mathbf{g} \in \mathbb{R}$ is defined so that

$$\mathbf{g}^T = \begin{pmatrix} g_1 & g_2 & \cdots & g_m \end{pmatrix}.$$

It follows that $\mathbf{p}^T = \mathbf{g}^T M^{-1}$, and therefore $\mathbf{p}^T M = \mathbf{g}^T$. Taking the ith component of the row vectors on both sides of this equality, we find that $\mathbf{p}^T \mathbf{u}^{(i)} = g_i$, as required. ∎

Lemma

Lemma STG-04 *Let m be a positive integer, let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ and $\hat{\mathbf{u}}^{(1)}, \hat{\mathbf{u}}^{(2)}, \ldots, \hat{\mathbf{u}}^{(m)}$ be bases of $\mathbb{R}^m$, and let $\mathbf{v}$ be an element of $\mathbb{R}^m$. Let h be an integer between 1 and m. Suppose that $\hat{\mathbf{u}}^{(h)} = \sum_{i=1}^{m} \mu_i \mathbf{u}^{(i)}$, where $\mu_1, \mu_2, \ldots, \mu_m$ are real numbers, and that $\mathbf{u}^{(i)} = \hat{\mathbf{u}}^{(i)}$ for all integers i between 1 and m satisfying $i \neq h$. Let $\mathbf{v} = \sum_{i=1}^{m} \lambda_i \mathbf{u}^{(i)} = \sum_{i=1}^{m} \hat{\lambda}_i \hat{\mathbf{u}}^{(i)}$. where $\lambda_i \in \mathbb{R}$ and $\hat{\lambda}_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$. Then*

$$\hat{\lambda}^{(i)} = \begin{cases} \dfrac{1}{\mu_h} \lambda_h & \text{if } i = h; \\ \lambda_i - \dfrac{\mu_i}{\mu_h} \lambda_h & \text{if } i \neq h. \end{cases}$$

Proof

From the representation of $\hat{\mathbf{u}}^{(h)}$ as a linear combination of $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(m)}$ we find that

$$\frac{1}{\mu_h}\hat{\mathbf{u}}^{(h)} = \mathbf{u}^{(h)} + \sum_{\substack{1 \le i \le m \\ i \ne h}} \frac{\mu_i}{\mu_h}\mathbf{u}^{(i)}.$$

Moreover $\hat{\mathbf{u}}^{(i)} = \mathbf{u}^{(i)}$ for all integers i between 1 and m satisfying $i \neq h$. It follows that

$$\mathbf{u}^{(h)} = \frac{1}{\mu_h}\hat{\mathbf{u}}^{(h)} - \sum_{\substack{1 \le i \le m \\ i \ne h}} \frac{\mu_i}{\mu_h}\hat{\mathbf{u}}^{(i)}.$$

Linear Algebra Results (continued)

It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \hat{\lambda}_i \hat{\mathbf{u}}^{(i)} &= \mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}^{(i)} \\
&= \sum_{\substack{1 \le i \le m \\ i \ne h}} \lambda_i \hat{\mathbf{u}}^{(i)} + \frac{1}{\mu_h} \lambda_h \hat{\mathbf{u}}^{(h)} - \sum_{\substack{1 \le i \le m \\ i \ne h}} \frac{\mu_i}{\mu_h} \lambda_h \mathbf{u}^{(i)}. \\
&= \sum_{\substack{1 \le i \le m \\ i \ne h}} \left(\lambda_i - \frac{\mu_i}{\mu_h} \lambda_h \right) \hat{\mathbf{u}}^{(i)} + \frac{1}{\mu_h} \lambda_h \hat{\mathbf{u}}^{(h)}
\end{aligned}$$

Linear Algebra Results (continued)

Therefore, equating coefficients of $\hat{\mathbf{u}}^{(i)}$ for $i = 1, 2, \ldots, n$, we find that

$$\hat{\lambda}^{(i)} = \begin{cases} \dfrac{1}{\mu_h}\,\lambda_h & \text{if } i = h, \\ \lambda_i - \dfrac{\mu_i}{\mu_h}\,\lambda_h & \text{if } i \neq h, \end{cases}$$

as required. ∎

The Extended Simplex Tableau for solving Linear Programming Problems

We now consider the construction of a tableau for a linear programming problem in Dantzig standard form. Such a problem is specified by an $m \times n$ matrix A, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. We suppose moreover that the matrix A is of rank m. We consider procedures for solving the following linear program in Danzig standard form.

> *Determine* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to the constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

We denote by $A_{i,j}$ the component of the matrix A in the ith row and jth column, we denote by b_i the ith component of the target vector $\mathbf{b}$ for $i = 1, 2, \ldots, m$, and we denote by c_j the jth component of the cost vector $\mathbf{c}$ for $j = 1, 2, \ldots, n$.

A *feasible* solution to this problem consists of an n-dimensional vector $\mathbf{x}$ that satisfies the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. An *optimal solution* to the problem is a feasible solution that minimizes the *objective function* $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the problem.

The Extended Simplex Tableau (continued)

A feasible solution to this optimization problem thus consists of real numbers $x_1, x_2, \ldots, x_n$ that satisfy the constraints

$$A_{i,1}x_1 + A_{i,2}x_2 + \cdots + A_{i,n}x_n = b_i$$

for $i = 1, 2, \ldots, m$, and

$$x_j \geq 0$$

for $j = 1, 2, \ldots, n$.

Such a feasible solution is optimal if and only if it minimizes the objective function

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

amongst all feasible solution to the problem.

We denote by $\mathbf{a}^{(j)}$ the m-dimensional column vector whose components are those in the jth column of the matrix A. A feasible solution of the linear programming problem then consists of non-negative real numbers $x_1, x_2, \ldots, x_n$ for which

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \mathbf{b}.$$

A feasible solution determined by $x_1, x_2, \ldots, x_n$ is optimal if it minimizes cost $\sum_{j=1}^{n} c_j x_j$ amongst all feasible solutions to the linear programming problem.

A *basis* B for the linear programming problem is a subset of $\{1, 2, \ldots, n\}$ with m elements which has the property that the vectors $\mathbf{a}^{(j)}$ for $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$.

In the following discussion we let

$$B = \{j_1, j_2, \ldots, j_m\},$$

where $j_1, j_2, \ldots, j_m$ are distinct integers between 1 and n.

We denote by M_B the invertible $m \times m$ matrix whose component $(M)_{i,k}$ in the ith row and jth column satisfies $(M_B)_{i,k} = (A)_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then the kth column of the matrix M_B is specified by the column vector $\mathbf{a}^{(j_k)}$ for $k = 1, 2, \ldots, m$, and thus the columns of the matrix M_B coincide with those columns of the matrix A that are determined by elements of the basis B.

The Extended Simplex Tableau (continued)

Every vector in $\mathbb{R}^m$ can be expressed as a linear combination of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$. It follows that there exist uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i\mathbf{a}^{(j_i)}.$$

It follows from Lemma STG-01 that

$$t_{i,j} = (M_B^{-1}\mathbf{a}^{(j)})_i \quad \text{and} \quad s_i = (M_B^{-1}\mathbf{b})_i$$

for $i = 1, 2, \ldots, m$.

The Extended Simplex Tableau (continued)

The standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$ is defined such that

$$(\mathbf{e}^{(k)})_i = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

It follows from Lemma STG-02 that

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k} \mathbf{u}^{(i)},$$

where $r_{i,k}$ is the coefficient $(M_B^{-1})_{i,k}$ in the ith row and kth column of the inverse M_B^{-1} of the matrix M_B.

The Extended Simplex Tableau (continued)

We can record the coefficients of the m-dimensional vectors

$$\mathbf{a}^{(1)},\ \mathbf{a}^{(2)}, \ldots,\ \mathbf{a}^{(n)},\ \mathbf{b},\ \mathbf{e}^{(1)},\ \mathbf{e}^{(2)}, \ldots,\ \mathbf{e}^{(m)}$$

with respect to the basis $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$, of $\mathbb{R}^m$ in a tableau of the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	.	.	$\cdots$	.	.	.	.	$\cdots$	.

The Extended Simplex Tableau (continued)

The definition of the quantities $t_{i,j}$ ensures that

$$t_{i,j_k} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

Also

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{i,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $i = 1, 2, \ldots, m$.

The Extended Simplex Tableau (continued)

If the quantities $s_1, s_2, \ldots, s_m$ are all non-negative then they determine a basic feasible solution $\mathbf{x}$ of the linear programming problem associated with the basis B with components $x_1, x_2, \ldots, x_n$, where $x_{j_i} = s_i$ for $i = 1, 2, \ldots, m$ and $x_j = 0$ for all integers j between 1 and n that do not belong to the basis B. Indeed

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \sum_{i=1}^{m} x_{j_i} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

The Extended Simplex Tableau (continued)

The *cost* C of the basic feasible solution $\mathbf{x}$ is defined to be the value $\overline{c}^T\mathbf{x}$ of the objective function. The definition of the quantities $s_1, s_2, \ldots, s_m$ ensures that

$$C = \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} c_{j_i} s_i.$$

If the quantities $s_1, s_2, \ldots, s_n$ are not all non-negative then there is no basic feasible solution associated with the basis B.

The Extended Simplex Tableau (continued)

The criterion row at the bottom of the tableau has cells to record quantities $p_1, p_2, \ldots, p_m$ associated with the vectors that constitute the standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$. These quantities are defined so that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$, where c_{j_i} is the cost associated with the basis vector $\mathbf{a}^{(j_i)}$ for $i = 1, 2, \ldots, k$, An application of Lemma STG-03 establishes that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = c_{j_i}$$

for $i = 1, 2, \ldots, k$.

The Extended Simplex Tableau (continued)

On combining the identities

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k, \quad p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k} \quad \text{and} \quad C = \sum_{i=1}^{m} c_{j_i} s_i$$

derived above, we find that

$$C = \sum_{i=1}^{m} c_{j_i} s_i = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{j_i} r_{i,k} b_k = \sum_{k=1}^{m} p_k b_k.$$

The Extended Simplex Tableau (continued)

The tableau also has cells in the criterion row to record quantities $-q_1, -q_2, \ldots, -q_n$, where $q_1, q_2, \ldots, q_n$ are the components of the unique n-dimensional vector $\mathbf{q}$ characterized by the following properties:

- $q_{j_i} = 0$ for $i = 1, 2, \ldots, m$;
- $\mathbf{c}^T \overline{\mathbf{x}} = C + \mathbf{q}^T \overline{\mathbf{x}}$ for all $\overline{\mathbf{x}} \in \mathbb{R}^m$ satisfying the matrix equation $A\overline{\mathbf{x}} = \mathbf{b}$.

The Extended Simplex Tableau (continued)

First we show that if $\mathbf{q} \in \mathbb{R}^n$ is defined such that $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ has the required properties.

The definition of $p_1, p_2, \ldots, p_k$ ensures (as a consequence of Lemma STG-03, as noted above) that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = c_{j_i}$$

for $i = 1, 2, \ldots, k$. It follows that

$$q_{j_i} = c_{j_i} - (\mathbf{p}^T A)_{j_i} = c_{j_i} - \sum_{k=1}^{m} p_k A_{k,j_i} = 0$$

for $i = 1, 2, \ldots, n$.

The Extended Simplex Tableau (continued)

Also $\mathbf{p}^T\mathbf{b} = C$. It follows that if $\overline{\mathbf{x}} \in \mathbb{R}^n$ satisfies $A\overline{\mathbf{x}} = \mathbf{b}$ then

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = C + \mathbf{q}^T\overline{\mathbf{x}}.$$

Thus if $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ satisfies the properties specified above.

The Extended Simplex Tableau (continued)

We next show that

$$(\mathbf{p}^T A)_j = \sum_{i=1}^{m} c_{j_i} t_{i,j}$$

for $j = 1, 2, \ldots, n$.

Now

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. (This is a consequence of the identities

$$\mathbf{a}^{(j)} = \sum_{k=1}^{m} A_{k,j} \mathbf{e}^{(k)} = \sum_{i=1}^{m} \sum_{k=1}^{m} r_{i,k} A_{k,j} \mathbf{a}^{(j_i)},$$

as noted earlier.)

Also the definition of p_k ensures that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$. These results ensure that

$$\sum_{i=1}^{m} c_{j_i} t_{i,j} = \sum_{i=1}^{m} \sum_{k=1}^{m} c_{j_i} r_{i,k} A_{k,j} = \sum_{k=1}^{m} p_k A_{k,j} = (\mathbf{p}^T A)_j.$$

It follows that

$$-q_j = \sum_{k=1}^{m} p_k A_{k,j} - c_j = \sum_{i=1}^{m} c_{j_i} t_{i,k} - c_j$$

for $j = 1, 2, \ldots, n$.

The Extended Simplex Tableau (continued)

The *extended simplex tableau* associated with the basis B is obtained by entering the values of the quantities $-q_j$ (for $j = 1, 2, \ldots, n$), C and p_k (for $k = 1, 2, \ldots, m$) into the bottom row to complete the tableau described previously. The extended simplex tableau has the following structure:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C	p_1	p_2	$\cdots$	p_m

Simplex Tableau Example (continued)

The extended simplex tableau can be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

Simplex Tableau Example (continued)

Let $\mathbf{c}_B$ denote the m-dimensional vector defined so that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{pmatrix}.$$

The identities we have verified ensure that the extended simplex tableau can therefore also be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \ldots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \ldots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1}A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$