# MA3484 Methods of Mathematical Economics
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## Simplex Tableau Example (continued)

We now carry through procedures for adjusting the basis and calculating the extended simplex tableau associated with the new basis.

We recall that the extended simplex tableau corresponding to the old basis $\{1, 2, 3\}$ is as follows:—

| | $\mathbf{a}^{(1)}$ | $\mathbf{a}^{(2)}$ | $\mathbf{a}^{(3)}$ | $\mathbf{a}^{(4)}$ | $\mathbf{a}^{(5)}$ | $\mathbf{b}$ | $\mathbf{e}^{(1)}$ | $\mathbf{e}^{(2)}$ | $\mathbf{e}^{(3)}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{a}^{(1)}$ | 1 | 0 | 0 | $-\frac{24}{23}$ | $-\frac{25}{23}$ | 1 | $-\frac{13}{23}$ | $\frac{4}{23}$ | $\frac{7}{23}$ |
| $\mathbf{a}^{(2)}$ | 0 | 1 | 0 | $\frac{27}{23}$ | $\frac{31}{23}$ | 3 | $\frac{6}{23}$ | $\frac{7}{23}$ | $-\frac{5}{23}$ |
| $\mathbf{a}^{(3)}$ | 0 | 0 | 1 | $\frac{13}{23}$ | $\frac{26}{23}$ | 2 | $\frac{8}{23}$ | $-\frac{6}{23}$ | $\frac{1}{23}$ |
| | 0 | 0 | 0 | $\frac{76}{23}$ | $\frac{60}{23}$ | 20 | $\frac{22}{23}$ | $\frac{18}{23}$ | $-\frac{3}{23}$ |

We now consider which of the indices 4 and 5 to bring into the basis.

Suppose we look for a basis which includes the vector $\mathbf{a}^{(4)}$ together with two of the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$. A feasible solution $\overline{\mathbf{x}}$ with $\overline{x}_5 = 0$ will satisfy

$$\overline{\mathbf{x}}^T = \left( \begin{array}{ccccc} 1 + \frac{24}{23}\lambda & 3 - \frac{27}{23}\lambda & 2 - \frac{13}{23}\lambda & \lambda & 0 \end{array} \right),$$

where $\lambda = \overline{x}_4$. Indeed $A(\overline{\mathbf{x}} - \mathbf{x}) = \mathbf{0}$, where $\mathbf{x}$ is the current basic feasible solution, and therefore

$$(\overline{x}_1 - 1)\mathbf{a}^{(1)} + (\overline{x}_2 - 3)\mathbf{a}^{(2)} + (\overline{x}_3 - 2)\mathbf{a}^{(3)} + \overline{x}_4\mathbf{a}^{(4)} = \mathbf{0}.$$

## Simplex Tableau Example (continued)

Now

$$\mathbf{a}^{(4)} = -\tfrac{24}{23}\mathbf{a}^{(1)} + \tfrac{27}{23}\mathbf{a}^{(2)} + \tfrac{13}{23}\mathbf{a}^{(3)},$$

It follows that

$$(\overline{x}_1 - 1 - \tfrac{24}{23}\overline{x}_4)\mathbf{a}^{(1)} + (\overline{x}_2 - 3 + \tfrac{27}{33}\overline{x}_4)\mathbf{a}^{(2)} + (\overline{x}_3 - 2 + \tfrac{13}{23}\overline{x}_4)\mathbf{a}^{(3)} = \mathbf{0}.$$

But the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ are linearly independent. Thus if $\overline{x}_4 = \lambda$ and $\overline{x}_5 = 0$ then

$$\overline{x}_1 - 1 - \tfrac{24}{23}\lambda = 0, \quad \overline{x}_2 - 3 + \tfrac{27}{23}\lambda = 0, \quad \overline{x}_3 - 2 + \tfrac{13}{23}\lambda = 0,$$

and thus

$$\overline{x}_1 = 1 + \tfrac{24}{23}\lambda, \quad \overline{x}_2 = 3 - \tfrac{27}{23}\lambda, \quad \overline{x}_3 = 2 - \tfrac{13}{23}\lambda.$$

# Simplex Tableau Example (continued)

For the solution $\overline{\mathbf{x}}$ to be feasible the components of $\overline{\mathbf{x}}$ must all be non-negative, and therefore $\lambda$ must satisfy

$$\lambda \leq \min\left(3 \times \tfrac{23}{27},\ \ 2 \times \tfrac{23}{13}\right).$$

Now $3 \times \frac{23}{27} = \frac{69}{27} \approx 2.56$ and $2 \times \frac{23}{13} = \frac{46}{13} \approx 3.54$. It follows that the maximum possible value of $\lambda$ is $\frac{69}{27}$. The feasible solution corresponding to this value of $\lambda$ is a basic feasible solution with basis $\{1, 3, 4\}$, and passing from the current basic feasible solution $\mathbf{x}$ to the new feasible basic solution would lower the cost by $-q_4\lambda$, where $-q_4\lambda = \frac{76}{23} \times \frac{69}{27} = \frac{228}{27} \approx 8.44$.

# Simplex Tableau Example (continued)

We examine this argument in more generality to see how to calculate the change in the cost that arises if an index $j$ not in the current basis is brought into that basis. Let the current basis be $\{j_1, j_2, j_3\}$. Then

$$\mathbf{b} = s_1 \mathbf{a}^{(j_1)} + s_2 \mathbf{a}^{(j_2)} + s_3 \mathbf{a}^{(j_3)}$$

and

$$\mathbf{a}^{(j)} = t_{1,j} \mathbf{a}^{(j_1)} + t_{2,j} \mathbf{a}^{(j_2)} + t_{3,j} \mathbf{a}^{(j_3)}.$$

Thus if $\overline{\mathbf{x}}$ is a feasible solution, and if $(\overline{\mathbf{x}})_{j'} = 0$ for $j' \notin \{j_1, j_2, j_3, j\}$, then

$$\overline{x}_{j_1} \mathbf{a}^{(j_1)} + \overline{x}_{j_2} \mathbf{a}^{(j_2)} + \overline{x}_{j_3} \mathbf{a}^{(j_3)} + \overline{x}_j \mathbf{a}^{(j)} - \mathbf{b} = \mathbf{0}.$$

Let $\lambda = \overline{x}_j$. Then

$$(\overline{x}_{j_1} + \lambda t_{1,j} - s_1)\mathbf{a}^{(j_1)} + (\overline{x}_{j_2} + \lambda t_{2,j} - s_2)\mathbf{a}^{(j_2)} + (\overline{x}_{j_3} + \lambda t_{3,j} - s_3)\mathbf{a}^{(j_3)} = \mathbf{0}.$$

But the vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_3)}$ are linearly independent, because $\{j_1, j_2, j_3\}$ is a basis for the linear programming problem. It follows that

$$\overline{x}_{j_i} = s_i - \lambda t_{i,j}$$

for $i = 1, 2, 3$.

# Simplex Tableau Example (continued)

For a feasible solution we require $\lambda \geq 0$ and $s_i - \lambda t_{i,j} \geq 0$ for $i = 1, 2, 3$. We therefore require

$$0 \leq \lambda \leq \min \left( \frac{s_i}{t_{i,j}} : t_{i,j} > 0 \right).$$

We could therefore obtain a new basic feasible solution by ejecting from the current basis an index $j_i$ for which the ratio $\dfrac{s_i}{t_{i,j}}$ has its minimum value, where this minimum is taken over those values of $i$ for which $t_{i,j} > 0$. If we set $\lambda$ equal to this minimum value, then the cost is then reduced by $-q_j \lambda$.

# Simplex Tableau Example (continued)

With the current basis we find that $s_2/t_{4,2} = \frac{69}{27}$ and $s_3/t_{4,3} = \frac{46}{13}$. Now $\frac{69}{27} < \frac{46}{13}$. It follows that we could bring the index 4 into the basis, obtaining a new basis $\{1, 3, 4\}$, to obtain a cost reduction equal to $\frac{228}{13}$, given that $\frac{76}{23} \times \frac{69}{27} = \frac{228}{13} \approx 8.44$.

We now calculate the analogous cost reduction that would result from bringing the index 5 into the basis. Now $s_2/t_{5,2} = \frac{69}{31}$ and $s_3/t_{5,3} = \frac{46}{26}$. Moreover $\frac{46}{26} < \frac{69}{31}$. It follows that we could bring the index 5 into the basis, obtaining a new basis $\{1, 2, 5\}$, to obtain a cost reduction equal to $\frac{60}{23} \times \frac{46}{26} = \frac{120}{26} \approx 4.62$.

We thus obtain the better cost reduction by changing basis to $\{1, 3, 4\}$.

## Simplex Tableau Example (continued)

We need to calculate the tableau associated with the basis $\{1, 3, 4\}$. We will initially ignore the change to the criterion row, and calculate the updated values in the cells of the other rows. The current tableau with the values in the criterion row deleted is as follows:—

| | $\mathbf{a}^{(1)}$ | $\mathbf{a}^{(2)}$ | $\mathbf{a}^{(3)}$ | $\mathbf{a}^{(4)}$ | $\mathbf{a}^{(5)}$ | $\mathbf{b}$ | $\mathbf{e}^{(1)}$ | $\mathbf{e}^{(2)}$ | $\mathbf{e}^{(3)}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{a}^{(1)}$ | 1 | 0 | 0 | $-\frac{24}{23}$ | $-\frac{25}{23}$ | 1 | $-\frac{13}{23}$ | $\frac{4}{23}$ | $\frac{7}{23}$ |
| $\mathbf{a}^{(2)}$ | 0 | 1 | 0 | $\frac{27}{23}$ | $\frac{31}{23}$ | 3 | $\frac{6}{23}$ | $\frac{7}{23}$ | $-\frac{5}{23}$ |
| $\mathbf{a}^{(3)}$ | 0 | 0 | 1 | $\frac{13}{23}$ | $\frac{26}{23}$ | 2 | $\frac{8}{23}$ | $-\frac{6}{23}$ | $\frac{1}{23}$ |
| | · | · | · | · | · | · | · | · | · |

# Simplex Tableau Example (continued)

Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$ and suppose that

$$\mathbf{v} = \mu_1 \mathbf{a}^{(1)} + \mu_2 \mathbf{a}^{(2)} + \mu_3 \mathbf{a}^{(3)} = \mu_1' \mathbf{a}^{(1)} + \mu_2' \mathbf{a}^{(4)} + \mu_3' \mathbf{a}^{(3)}.$$

Now

$$\mathbf{a}^{(4)} = -\tfrac{24}{23} \mathbf{a}^{(1)} + \tfrac{27}{23} \mathbf{a}^{(2)} + \tfrac{13}{23} \mathbf{a}^{(3)}.$$

On multiplying this equation by $\frac{23}{27}$, we find that

$$\tfrac{23}{27} \mathbf{a}^{(4)} = -\tfrac{24}{27} \mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \tfrac{13}{27} \mathbf{a}^{(3)},$$

and therefore

$$\mathbf{a}^{(2)} = \tfrac{24}{27} \mathbf{a}^{(1)} + \tfrac{23}{27} \mathbf{a}^{(4)} - \tfrac{13}{27} \mathbf{a}^{(3)}.$$

# Simplex Tableau Example (continued)

It follows that

$$\mathbf{v} = (\mu_1 + \tfrac{24}{27}\mu_2)\mathbf{a}^{(1)} + \tfrac{23}{27}\mu_2\mathbf{a}^{(4)} + (\mu_3 - \tfrac{13}{27}\mu_2)\mathbf{a}^{(3)},$$

and thus

$$\mu_1' = \mu_1 + \tfrac{24}{27}\mu_2, \quad \mu_2' = \tfrac{23}{27}\mu_2, \quad \mu_3' = \mu_3 - \tfrac{13}{27}\mu_2.$$

# Simplex Tableau Example (continued)

Now each column of the tableau specifies the coefficients of the vector labelling the column of the tableau with respect to the basis specified by the vectors labelling the rows of the tableau.

The *pivot row* of the old tableau is that labelled by the vector $\mathbf{a}^{(2)}$ that is being ejected from the basis. The *pivot column* of the old tableau is that labelled by the vector $\mathbf{a}^{(4)}$ that is being brought into the basis. The *pivot element* of the tableau is the element or value in both the pivot row and the pivot column. In this example the pivot element has the value $\frac{27}{23}$.

## Simplex Tableau Example (continued)

We see from the calculations above that the values in the pivot row of the old tableau are transformed by multiplying them by the reciprocal $\frac{23}{27}$ of the pivot element; the entries in the first row of the old tableau are transformed by adding to them the entries below them in the pivot row multiplied by the factor $\frac{24}{27}$; the values in the third row of the old tableau are transformed by subtracting from them the entries above them in the pivot row multiplied by the factor $\frac{13}{27}$.

# Simplex Tableau Example (continued)

Indeed the coefficients $t_{i,j}$, $s_i$, $r_{i,k}$, $t'_{i,j}$, $s'_i$ and $r'_{i,k}$ are defined for $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$ and $k = 1, 2, 3$ so that

$$\begin{aligned} \mathbf{a}^{(j)} &= \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} t'_{i,j}\mathbf{a}^{(j'_i)}, \\ \mathbf{b} &= \sum_{i=1}^{3} s_i\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} s'_i\mathbf{a}^{(j'_i)}, \\ \mathbf{e}^{(k)} &= \sum_{i=1}^{3} r_{i,k}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} r'_{i,k}\mathbf{a}^{(j'_i)}, \end{aligned}$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$.

# Simplex Tableau Example (continued)

The general rule for transforming the coefficients of a vector when changing from the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ to the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(4)}, \mathbf{a}^{(3)}$ ensure that

$$\begin{aligned} t'_{2,j} &= \frac{1}{t_{2,4}}\, t_{2,j}, \\ t'_{i,j} &= t_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, t_{2,j} \quad (i = 1, 3). \\ s'_2 &= \frac{1}{t_{2,4}}\, s_2, \\ s'_i &= s_i - \frac{t_{i,4}}{t_{2,4}}\, s_2 \quad (i = 1, 3). \end{aligned}$$

## Simplex Tableau Example (continued)

$$\begin{aligned} r'_{2,k} &= \frac{1}{t_{2,4}}\, r_{2,j}, \\ r'_{i,k} &= r_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, r_{2,j} \quad (i = 1, 3). \end{aligned}$$

The quantity $t_{2,4}$ is the value of the pivot element of the old tableau. The quantities $t_{2,j}$, $s_2$ and $r_{2,k}$ are those that are recorded in the pivot row of that tableau, and the quantities $t_{i,4}$ are those that are recorded in the pivot column of the tableau.

## Simplex Tableau Example (continued)

We thus obtain the following tableau:–

| | $\mathbf{a}^{(1)}$ | $\mathbf{a}^{(2)}$ | $\mathbf{a}^{(3)}$ | $\mathbf{a}^{(4)}$ | $\mathbf{a}^{(5)}$ | $\mathbf{b}$ | $\mathbf{e}^{(1)}$ | $\mathbf{e}^{(2)}$ | $\mathbf{e}^{(3)}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{a}^{(1)}$ | 1 | $\frac{24}{27}$ | 0 | 0 | $\frac{3}{27}$ | $\frac{99}{27}$ | $-\frac{9}{27}$ | $\frac{12}{27}$ | $\frac{3}{27}$ |
| $\mathbf{a}^{(4)}$ | 0 | $\frac{23}{27}$ | 0 | 1 | $\frac{31}{27}$ | $\frac{69}{27}$ | $\frac{6}{27}$ | $\frac{7}{27}$ | $-\frac{5}{27}$ |
| $\mathbf{a}^{(3)}$ | 0 | $-\frac{13}{27}$ | 1 | 0 | $\frac{13}{27}$ | $\frac{15}{27}$ | $\frac{6}{27}$ | $-\frac{11}{27}$ | $\frac{4}{27}$ |
| | . | . | . | . | . | . | . | . | . |

# Simplex Tableau Example (continued)

The values in the column of the tableau labelled by the vector **b** give us the components of a new basic feasible solution $\mathbf{x}'$. Indeed the column specifies that

$$\mathbf{b} = \tfrac{99}{27}\mathbf{a}^{(1)} + \tfrac{69}{27}\mathbf{a}^{(4)} + \tfrac{15}{27}\mathbf{a}^{(2)},$$

and thus $A\mathbf{x}' = \mathbf{b}$ where

$$\mathbf{x}'^T = \begin{pmatrix} \frac{99}{27} & 0 & \frac{15}{27} & \frac{69}{27} & 0 \end{pmatrix}.$$