MA3484 Methods of Mathematical Economics
School of Mathematics, Trinity College
Hilary Term 2015
Lecture 17 (February 19, 2015)

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Simplex Method Example

We consider again the following linear programming problem.

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:

$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11;$

$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6;$

$x_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$

Simplex Method Example (continued)

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 satisfying the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Simplex Method Example (continued)

Let

$$\mathbf{a}^{(1)} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(5)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

For a feasible solution to the problem we must find non-negative real numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + x_3\mathbf{a}^{(3)} + x_4\mathbf{a}^{(4)} + x_5\mathbf{a}^{(5)} = \mathbf{b}.$$

An optimal solution to the problem is a feasible solution that minimizes

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5$$

amongst all feasible solutions to the problem, where $c_1 = 3$, $c_2 = 4$, $c_3 = 2$, $c_4 = 9$ and $c_5 = 5$.

Simplex Method Example (continued)

Let $\mathbf{c}$ denote the column vector whose ith component is c_i respectively. Then

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix},$$

and an optimal solution is a feasible solution that minimizes $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the problem. We refer to the quantity $\mathbf{c}^T\mathbf{x}$ as the *cost* of the feasible solution $\mathbf{x}$.

Simplex Method Example (continued)

Let $I = \{1, 2, 3, 4, 5\}$. A *basis* for this optimization problem is a subset $\{j_1, j_2\}$ of I, where $j_1 \neq j_2$, for which the corresponding vectors $\mathbf{a}^{j_1}, \mathbf{a}^{j_2}$ constitute a basis of $\mathbb{R}^2$. By inspection we see that each pair of vectors taken from the list $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}$ consists of linearly independent vectors, and therefore each pair of vectors from this list constitutes a basis of $\mathbb{R}^2$. It follows that every subset of I with exactly two elements is a basis for the optimization problem.

A feasible solution $(x_1, x_2, x_3, x_4, x_5)$ to this optimization problem is said to be a *basic feasible solution* if there exists a basis B for the optimization problem such that $x_j = 0$ when $j \neq B$.
In the case of the present problem, all subsets of $\{1, 2, 3, 4, 5\}$ with exactly two elements are bases for the problem. It follows that a feasible solution to the problem is a *basic feasible solution* if and only if the number of non-zero components of the solution does not exceed 2.

We take as given the following initial basic feasible solution $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$. One can readily verify that $\mathbf{a}^{(1)} + 2\mathbf{a}^{(2)} = \mathbf{b}$. This initial basic feasible solution is associated with the basis $\{1, 2\}$. The cost of this solution is 11.

Simplex Method Example (continued)

We now set out to explore the design of tableaux to organize the calculation of optimal solutions to linear programming problems like that under discussion. We denote by $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ the standard basis of $\mathbb{R}^2$, where

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ be a basis of $\mathbb{R}^2$. Then there exists an invertible matrix M such that $\mathbf{u}^{(1)} = M\mathbf{e}^{(1)}$ and $\mathbf{u}^{(2)} = M\mathbf{e}^{(2)}$.

Simplex Method Example (continued)

Moreover if

$$\mathbf{u}^{(1)} = \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix}, \quad \mathbf{u}^{(2)} = \begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix},$$

then

$$M = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} \\ u_2^{(1)} & u_2^{(2)} \end{pmatrix}$$

and

$$M^{-1} = \frac{1}{u_1^{(1)} u_2^{(2)} - u_2^{(1)} u_1^{(2)}} \begin{pmatrix} u_2^{(2)} & -u_1^{(2)} \\ -u_2^{(1)} & u_1^{(1)} \end{pmatrix}.$$

Simplex Method Example (continued)

Let $\mathbf{v}$ be an element of $\mathbb{R}^2$. Then there exist real numbers λ_1 and λ_2 such that $\mathbf{v} = \lambda_1 \mathbf{u}^{(1)} + \lambda_2 \mathbf{u}^{(2)}$. Then

$$\begin{aligned} M^{-1}\mathbf{v} &= \lambda_1 M^{-1}\mathbf{u}^{(1)} + \lambda_2 M^{-1}\mathbf{u}^{(2)} = \lambda_1 \mathbf{e}^{(1)} + \lambda_2 \mathbf{e}^{(2)} \\ &= \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \end{aligned}$$

It follows in particular that

$$\begin{aligned} \mathbf{e}^{(1)} &= (M^{-1})_{1,1}\mathbf{u}^{(1)} + (M^{-1})_{2,1}\mathbf{u}^{(2)}, \\ \mathbf{e}^{(2)} &= (M^{-1})_{1,2}\mathbf{u}^{(1)} + (M^{-1})_{2,2}\mathbf{u}^{(2)}. \end{aligned}$$

Simplex Method Example (continued)

Moreover if

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

then

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2,$$

where

$$\begin{aligned} \lambda_1 &= (M^{-1})_{1,1} v_1 + (M^{-1})_{1,2} v_2 \\ \lambda_2 &= (M^{-1})_{2,1} v_1 + (M^{-1})_{2,2} v_2 \end{aligned}$$

Simplex Method Example (continued)

In particular

$$\mathbf{a}^{(j)} = t_{1,j}\mathbf{u}^{(1)} + t_{2,j}\mathbf{u}^{(2)}$$

for $j = 1, 2, \ldots, n$, where

$$\begin{pmatrix} t_{1,j} \\ t_{2,j} \end{pmatrix} = M^{-1}\mathbf{a}^{(j)}.$$

Also

$$\mathbf{b} = s_1\mathbf{u}^{(1)} + s_2\mathbf{u}^{(2)}$$

where

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = M^{-1}\mathbf{b}.$$

We can then set up a tableau where the rows are labelled by the basis elements $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, the columns are labelled by the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, $\mathbf{a}^{(3)}$, $\mathbf{a}^{(4)}$, $\mathbf{a}^{(5)}$, $\mathbf{b}$, $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, and where the cells of the tableau contain the components of the vector labelling the column with respect to the basis vector labelling the row.

Simplex Method Example (continued)

This tableau will thus take the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{u}^{(1)}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	s_1	$(M^{-1})_{1,1}$	$(M^{-1})_{1,2}$
$\mathbf{u}^{(2)}$	$t_{2,1}$	$t_{2,2}$	$t_{2,3}$	$t_{2,4}$	$t_{2,5}$	s_2	$(M^{-1})_{2,1}$	$(M^{-1})_{2,2}$

Simplex Method Example (continued)

Now our initial basic solution $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$ is associated with a basis B, where $B = \{1, 2\}$. This basis determines a basis $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ of $\mathbb{R}^2$, where $\mathbf{u}^{(1)} = \mathbf{a}^{(1)}$ and $\mathbf{u}^{(2)} = \mathbf{a}^{(2)}$. Then $M = M_B$, where M_B is the matrix determined by the first two columns of the matrix A. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}, \quad M_B^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

Simplex Method Example (continued)

Now

$$
\begin{aligned}
M_B^{-1}\mathbf{a}^{(1)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
M_B^{-1}\mathbf{a}^{(2)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
M_B^{-1}\mathbf{a}^{(3)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ \frac{1}{7} \end{pmatrix}, \\
M_B^{-1}\mathbf{a}^{(4)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{17}{7} \\ -\frac{12}{7} \end{pmatrix},
\end{aligned}
$$

Simplex Method Example (continued)

$$
\begin{aligned}
M_B^{-1}\mathbf{a}^{(5)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ -\frac{8}{7} \end{pmatrix}, \\
M_B^{-1}\mathbf{b} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 11 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \\
M_B^{-1}\mathbf{e}^{(1)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} \\ \frac{4}{7} \end{pmatrix}. \\
M_B^{-1}\mathbf{e}^{(2)} &= -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ -\frac{5}{7} \end{pmatrix}.
\end{aligned}
$$

Entering these values into the tableau yields the following completed tableau:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{u}^{(1)}$	1	0	$\frac{5}{7}$	$\frac{17}{7}$	$\frac{9}{7}$	1	$-\frac{1}{7}$	$\frac{3}{7}$
$\mathbf{u}^{(2)}$	0	1	$\frac{1}{7}$	$-\frac{12}{7}$	$-\frac{8}{7}$	2	$\frac{4}{7}$	$-\frac{5}{7}$

The basic rule governing this tableau is that the vector heading each each should be the linear combination of the vectors heading the two rows, with coefficients taken from the relevant column.

Simplex Method Example (continued)

Suppose we wish to replace $\mathbf{u}^{(1)}$ by $\mathbf{u}^{(3)}$ in the basis. Now $\mathbf{u}^{(3)} = \frac{5}{7}\mathbf{u}^{(1)} + \frac{1}{7}\mathbf{u}^{(2)}$, and therefore $\mathbf{u}^{(1)} = \frac{7}{5}\mathbf{u}^{(3)} - \frac{1}{5}\mathbf{u}^{(2)}$. Accordingly we subtract from the values in the second row one fifth of the values in the first row. We then multiply the values in the first row by $\frac{7}{5}$ and relabel the first row by $\mathbf{a}^{(3)}$. We obtain in this fashion the following tableau:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{u}^{(3)}$	$\frac{7}{5}$	0	1	$\frac{17}{5}$	$\frac{9}{5}$	$\frac{7}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$
$\mathbf{u}^{(2)}$	$-\frac{1}{5}$	1	0	$-\frac{11}{5}$	$-\frac{7}{5}$	$\frac{9}{5}$	$\frac{3}{5}$	$-\frac{4}{5}$